**Applied Mathematics, Ordinary Differential Equations**
**Preliminary Exam—January 2024**

# "Can one hear the shape of a drum"

This problem explores Marc Kac's famous question from 1966 [1]: "Can one hear the shape of a drum?" in other words, can we guess the shape of the drum by listening to its sound? While the problem has limited engineering interest, it raises the fundamental scientific question of how much information we can learn about a continuous system from a finite or countable set of measurements.

We will focus on the forward problem and reconstruct the sound of a vibrating drum from its vibration in simple one- and two-dimensional settings. To do so, we will assume the vibrations $u(\mathbf{x}, t)$ of a drum of shape $\Omega$ satisfy the following differential equation

$$\frac{\partial^2 u(\mathbf{x}, t)}{\partial t^2} = \triangle u(\mathbf{x}, t) \qquad \forall \mathbf{x} \in \Omega, \quad \forall t \geq 0, \tag{1}$$

where $\triangle$ denotes the Laplacian operator ($\triangle = \partial_{xx} + \partial_{yy}$ in 2D Cartesian coordinates). We will assume that the contour $\partial\Omega$ of the membrane is fixed, and so

$$u(\mathbf{x}, t) = 0 \qquad \forall \mathbf{x} \in \partial\Omega, \quad \forall t \geq 0. \tag{2}$$

Additionally, we will assume that the vibrations remain bounded for all time.

As you will see, this *a priori* complicated multidimensional problem can easily be transformed into standard ODE problems using simple mathematical tricks and thus studied using the theory you learned and reviewed during your preparation.

Within each part, questions are often largely independent. Not answering a question will not necessarily prevent you from answering the following ones. You are allowed to use any given results you could not prove.

## 1 Analysis

1. Classify the above equation (1)(order, ordinary/partial, autonomous, homogeneous, linear/non-linear).
2. Assuming that we work in 1D and that the membrane $\Omega$ is the segment $[0, L]$, sketch the problem: in particular, draw the segment, sketch the membrane vibrations as a function of $\mathbf{x}$ over the segment and at any given time. Make sure that"your sketch satisfies the boundary condition (2)."
3. Assume now that the membrane is a two-dimensional disk of radius L. Sketch the problem.

## 2 Separation of Variables

For the entire exam, we will seek separable solutions of the equation (1) of the form

$$u(\mathbf{x}, t) = X(\mathbf{x})T(t). \tag{3}$$

Doing so, we can transform the multivariable solution into the product of single variable functions and ultimately work with ordinary differential equations only.

1. Assuming that $u$ has the form (3), express $\frac{\partial u}{\partial t}$ and $\triangle u$ in term of $X, T$ and their derivatives.
2. Prove that if the solution has the above form (3) and is not uniformly equal to zero

$$\frac{T''(t)}{T(t)} = \frac{\triangle X(\mathbf{x})}{X(\mathbf{x})}. \tag{4}$$

3. Evaluate the above equation at any given position $\mathbf{x_0}$ to remark that the left-hand side ratio must be constant. Conclude that the above ratios must be equal to a constant $c \in \mathbb{R}$, and in particular that

$$T''(t) - cT(t) = 0. \tag{5}$$

4. Find the general solution of the above ODE (5) for both cases
   - $c > 0$
   - $c < 0$
5. Justify why $c > 0$ is unrealistic. We will define $c = -k^2$.
6. Show that $T(t)$ oscillates, and specify the frequency/note of its oscillations.
7. Rewriting (4) as

$$\triangle X = -k^2 X, \tag{6}$$

   what is the tuple $(-k^2, X)$ for the linear operator $\triangle$? (hint: think of $X$ as a vector in a function space) Do you expect $k$ to be uniquely defined and thus to find only one separable solution?
8. **Extra credit:** What is dictating the frequencies/notes produced by a drum?

## 3 1D drum - Vibrating String

For this part, we will assume the membrane $\Omega$ to be the segment $[0, L]$ and keep seeking separable solutions of the form (3). Since the variable $\mathbf{x}$ has only one component (x), we will write it as x. Again, we will define $c = -k^2$.

1. Explain why $X(0) = X(L) = 0$.
2. Explain why the expression you found for $T(t)$ in the previous part is still valid.
3. Justify why

$$X''(x) + k^2 X(x) = 0. \tag{7}$$

4. Find the general solution of the above equation.
5. Using the boundary conditions from question 1, prove that if $u(x,t)$ is a separable solution of the form (3), then the constant $k$ must be of the form

$$k_n = \frac{n\pi}{L}, \qquad n \in \mathbb{N}. \tag{8}$$

6. How many separable solutions have you found?

7. What note (i.e., temporal frequency) is associated with each separable solution?

8. Verify that the trigonometric series

$$\sum_{n=0}^{N} \left(A_n \cos(k_n t) + B_n \sin(k_n t)\right) \sin(k_n x) \tag{9}$$

is a solution of the wave equation for any coefficients $A_n, B_n$. What happens as $N \to \infty$? Is the trigonometric series guaranteed to converge? For it to converge, what is a necessary condition on $A_n, B_n$?

9. We will assume that the general solution of the wave equation can be written in the form

$$u(x,t) = \sum_{n=0}^{\infty} \left(A_n \cos(k_n t) + B_n \sin(k_n t)\right) \sin(k_n x), \tag{10}$$

where the coefficients $A_n, B_n$ depend on the initial perturbation of the string. We will assume that from a recording of the sound produced by the string, you can extract its temporal frequencies $k_n$ and corresponding coefficients $A_n, B_n$ using signal processing theory and Fast Fourier Transforms. From the sound it produces, can you figure out the length of the string? Can you hear the shape of the drum?

## 4 2D drum - Circular Membrane

We now consider $\Omega$ to be the disk of radius $L$ and center 0. We will use cylindrical coordinates $(r, \theta)$. In this coordinate system, the Laplacian of $u(r,\theta)$ takes the form

$$\triangle u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \tag{11}$$

and equation (1) becomes

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \qquad \forall r \in [0, L], \quad \forall \theta \in [0, 2\pi]. \tag{12}$$

Since we know the temporal part $T(t)$ of the solution from the first part of the problem, we will only focus on the spatial component $X(\mathbf{x})$ and further separate it as

$$X(\mathbf{x}) = R(r)\Theta(\theta). \tag{13}$$

Note that $\Theta$ is $2\pi$ periodic.

1. Justify why $R(L) = 0$.

2. Prove that

$$-\frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} = \frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + r^2k^2, \tag{14}$$

where, again, $k^2 = -c$. (hint: you may want to start directly from (4) or (6) and use expression (11).)

3. Explain why there exists a real constant $\alpha$ such that

$$-\frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} = \alpha. \tag{15}$$

4. Find the general solution of the above equation and explain why $\alpha$ must be positive. From now on, we set $\alpha = \nu^2$.

5. Explain why $\nu$ must be of the form

$$\nu_l = l\pi, \quad l \in \mathbb{N}. \tag{16}$$

6. Prove that

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + \left(k^2 - \frac{v^2}{r^2}\right) R = 0. \tag{17}$$

7. We seek to construct the general solution of the above equation (17) as a power series of the form

$$R(r) = \sum_{n=0}^{\infty} a_n r^{\sigma+n}, \tag{18}$$

where $\sigma \in \mathbb{R}$ is a parameter yet to be determined.

   (a) Plug in the above power series solution into (17) and obtain a power series equation.
   (b) Focusing on the zeroth order term ($n = 0$) of you equation, prove that $\sigma = \pm\nu$, and explain why $\sigma = -\nu$ leads to an unrealistic solution. From now on, we will focus on the case $\sigma = \nu$.
   (c) Looking now at the first order term ($n = 1$), prove that $a_1 = 0$.
   (d) Prove that

$$a_n = -a_{n-2}\frac{k^2}{n(n+2\nu)} \tag{19}$$

   and explain why all odd terms are equal to zero.
   (e) Prove that the power series solution has an infinite radius of convergence.

8. For the rest of the problem, we will admit that the general solution of (17) can be expressed as

$$R(r) = C\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu+n+1)n!}\left(\frac{rk}{2}\right)^{\nu+2n}, \tag{20}$$

where $C$ is a constant coefficient, and $\Gamma(z)$ is the gamma function

$$\Gamma(z) = \int_0^{\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t, \tag{21}$$

such that

$$\Gamma(n) = (n-1)! \quad \forall n \in \mathbb{N} \tag{22}$$

Explain why

$$0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu + n + 1)n!} \left(\frac{Lk}{2}\right)^{2n}. \tag{23}$$

(hint: you may want to look back at the very beginning of this part)

9. This last equation should be interpreted as a relation between $k$ and $\nu$. Since we know that $\nu$ must have a specific form (see (16)), this equation tells us the admissible values of $k$ for any particular value of $\nu$. Since the power series converges fast, we will truncate it to investigate the connection between $k$ and $\nu$. For this question, we assume that $\nu$ is fixed and equal to $\nu_l = l\pi$.
   (a) Linearize condition (23) to retain only the first two non-zero terms.
   (b) Solve your linearized equation for k. There is no need to be explicit about the gamma function; you can keep $\Gamma(\nu + 1)$, etc ... in your expression.
   (c) Keeping the first three non-zero terms, how is the expression for k affected?
   (d) For $\nu = 0$, estimate the admissible values of k.
10. What frequencies/notes will a circular drum produce?
11. **extra credit:** As an applied mathematician, living in a world where drums are two-dimensional and circular, with everything you have learned and all the signal processing tools you want ...

*Can you hear the shape of a drum?*

## References

[1] M. Kac. Can one hear the shape of a drum? *The American Mathematical Monthly*, 73(4):1–23, 1966.