Differential Equations Preliminary Exam *Jan 2025*

Duration: 240 minutes

Instructions: Answer all questions, without the use of notes, books or calculators, other than one handwritten sheet of your own notes. Partial credit will be awarded for correct work, unless otherwise specified. The total number of points is 70.

1. (12pts: 4,4,4) Given the autonomous system $\frac{dy}{dt} = (y^2 - 2y + 1)(y^2 + 3y)$
   (a) Find all the equilibrium points
   (b) Determine the stability of each equilibrium point
   (c) Sketch trajectories of solutions in the ty-plane, making sure to be consistent with your answers from (a) and (b).

2. (6pts) Give the general solution to
$$\frac{dy}{dx} = (x+2)y^2 + (x+2)$$

3. (6pts) Give the solution to the Initial Value Problem $y' + 2xy = x$ with $y(0) = 1$.

4. (7pts) Give a general solution to
$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 12t + 7e^{-2t}$$

5. (10pts) Use the Frobenius method to find the general solution to
$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = xe^x.$$
   You may leave expressions for coefficients in recursive form, but be sure to indicate how the recursions begin.

6. (10pts) Give a two-dimensional linear system of differential equations for which the general solution is a saddle point centered at the origin with stable direction $\vec{v}_1$ and unstable direction $\vec{v}_2$, with
$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

7. (9pts) Solve the following Boundary Value Problem, stating clearly for which values of $\beta \in \mathbb{R}$ you get two solutions, a unique solution, no solution, or infinitely many solutions.
$$y'' + 2y' + 4y = \beta y \quad \text{with } y(0) = 1 \text{ and } \lim_{t\to\infty} y(t) \text{ is finite}$$

8. (10pts: 6,4) The population of seals in Lake Baikal, $\vec{s}(t)$, can be modeled as having a number of juveniles, $s_1(t)$, a number of adults, $s_2(t)$ and a number of elders, $s_3(t)$.
   (a) Assuming limitless resources and that only adults may have offsprings, give a linear model, $\frac{d\vec{s}}{dt}$ =A $\vec{s}$, for those three populations over time (with time t counted in generations). A generation is meant as the time required for juveniles to become adults, and adults to become elders. You may use generic coefficients, c_1, c_2, etc, but assume that all $c_i > 0$. Give an interpretation for each parameter you introduce.
   (b) What are the possible long-term outcomes for the population of seals? Give conditions on A for each possible outcome but do not compute anything based on your specific model.

   (Bonus) Draw Lake Baikal and its seals.