Applied Mathematics Preliminary Exam Instructions

The exams will be given as follows:
Jan 11 (9:00 am- 1:00 pm): Advanced Calculus
Jan 12 (9:00 am- 1:00 pm): ODEs
Jan 13 (9:00 am- 1:00 pm): Linear Algebra

Read the following instructions carefully:

- A proctor will be available to help with any issues and questions. They will have access to contact both Shilpa Khatri and the professor who wrote the exam.
- Academic integrity is the foundation of an academic community and without it none of the educational or research goals of the university can be achieved. Academic integrity applies to research as well as undergraduate and graduate coursework/exams. Existing policies forbid cheating on examinations, plagiarism and other forms of academic dishonesty. UC Merced students are held to high standards of personal and professional conduct in compliance with the UC Merced Academic Honesty Policy and the UCM Code of Student Conduct. UCM Code of Student Conduct can be found here: http://studentconduct.ucmerced.edu
- Please include as the first page of your exam solutions a copy of following statement and your signature: "By completing this assignment/exam, I acknowledge and confirm that I will not give or receive any unauthorized assistance on this assignment/examination. I will conducted myself within the guidelines of the university academic integrity guidelines."
- These are closed-book written examinations. You are allowed one 8.5 x 11 inch sheet of paper (two-sided) that you can write whatever you want on it; only hand written material is allowed (no copy machine or computer reducing)
- Write each problem on a separate page.
- It is important to show your work for each problem. Credit will NOT be given for correct answers without justification. Also, partial credit will be given for incorrect answers if some of the work is correct.
- Clearly mark out (cross out) any work that you are not including in your answer and you do not want graded.

Applied Mathematics, Ordinary Differential Equations
Preliminary Exam—January 2022

Instructions and advice:

- For each problem, read carefully every single question. Then, read carefully every single question again.
- Within each part, questions are often largely independent. Not answering a question will not necessarily prevent you from answering the following ones.
- Make sure to justify all your answers.
- Good luck!

Part I - Socio-ecological problem

We are interested in modelling the impact of the public environmental concern on the environment. To do so we consider the coupled system of ODE

$$\begin{aligned} x' &= rx(1-x) - pxy, \qquad x(0) = x_0 \\ y' &= x(1-y), \qquad y(0) = y_0 \end{aligned}$$

- $x(t)$ represents the damage to the environment at time t: if $x = 0$ the environment is in perfect condition, if $x = 1$ the environment is completely ruined.
- $y(t)$ represents the public concern over the environment at tie t: $y = 0$ means that nobody cares about the environment. $y = 1$ means that everyone is concerned about the environment.
- We will assume that both x and y remain between 0 and 1.
- $p, r \in \mathbb{R}$ are two parameters

Simple model

In this first question we assume that nobody cares about the environement and so $y(t) = 0 \quad \forall t \in \mathbb{R}$. We therefore focus on the following Initial Value Problem (IVP) for $x(t)$

$$\dot{x} \ = rx(1-x), \qquad x(0) = x_0 \tag{1}$$

1. Prove that the above IVP has a unique solution for any initial condition x_0 and all parameter r.
2. How do you interpret $r > 0$? (hint: remember $0 \leq x(t) \leq 1$...)
3. From now on we set $r = 1$. Sketch the phase diagram for $(x, \dot{x})$.
4. From your sketched diagram, what is the limit of $x(t)$ as $t \to \infty$?
5. Using the method of your choice, solve the above IVP and find x(t).
6. If nobody cares, what happen to the environment?

Two-dimensional system

We now consider that the public is concerned about the environment and therefore $x(t)$ is no longer uniformly equal to zero, and consider the full coupled system. We still asssume that $r = 1$.

1. How do you interpret $p > 0$? For the rest of the problem we will set $p = \frac{1}{2}$.
2. rewrite the system in the compact form
$$\dot{X} = F(X), \qquad X(0) = X_0, \qquad X(t) \in \mathbb{R}. \tag{2}$$
3. Prove that any point on the $y-$axis (i.e. x=0) is stationary.
4. Prove that is only stationary point away from the $y-$axis. For the rest of the problem we will call this point A.
5. Compute the Jacobian of $F(X)$, and explain why all stationary point on the $y-$axis are unstable (hint: remember $0 \leq y(t) \leq 1$.
6. Determine whether A is stable or unstable by looking at the eigenvalues of the linearized problem around A.
7. Sketch the full phase diagram for $0 \leq x(t) \leq 1$ and $0 \leq y(t) \leq 1$.
8. Interpret your phase diagram, and describe the impact of the public concern over the environment. Can the environment ever be perfectly clean?

Realistic model

In this last part, we introduce $z(t)$ which represent the public money spent to clean up the environment. The new coupled system is

$$\begin{aligned} \dot{x} &= x(1 - x - \frac{y}{2} - \frac{z}{4}), &(3)\\ \dot{y} &= x(1 - y), &(4)\\ \dot{z} &= xy(1 - z). &(5) \end{aligned}$$

We will assume that $z(t) \geq 0$.

1. How do you interpret the $\frac{z}{4}$ and $xy(1 - z)$ terms?
2. Verify that $B = (\frac{1}{4}, 1, 1)$ is a stationary point.
3. What are the eigenvalues of the Jacobian evaluated at B? Determine whether B is stable or unstable.
4. Assuming that the initial condition X_0 is close to the stationary point B, describe the long term impact of public money spent on the environment.

Part II - Power Series: ~~The good,~~ the bad and the ugly

The bad

Consider the initial value problem

$$yy'' + y'^2 = 0, \qquad y(0) = 1, \qquad y'(0) = 1. \tag{6}$$

We assume that $y(x)$ can be represented as a power series centered at $x = 0$, and therefore there exists a sequence of coefficients a_n such that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{7}$$

We denote by $y_0^{(n)}$ the n^{th} derivative of y evaluated at $x = 0$.

1. Explain why finding the coefficients a_n might be challenging.
2. Using the Taylor expansion, find a relations between the coefficients a_n and the derivatives of y.
3. Using the initial conditions, find $y_0^{(2)}$. Differentiate the ODE to find an expression for $y_0^{(3)}$, $y_0^{(4)}$ and $y_0^{(5)}$.
4. What are the first six coefficients of the power series solution?
5. What does the radius of convergence seem to be?

The ugly

Consider the initial value problem

$$y'' + xy' + x^2 y = 0, \qquad y(0) = 1, \qquad y'(0) = 1. \tag{8}$$

1. Explain why seeking a power series solution is in this case a good strategy.
2. Find the the power series solution of the above IVP.
3. What can you say about its radius of convergence?

Part III -Odd Differential Equations

Renaming the constant

Find all functions $f(x)$ such that

$$f'(x) + f(x) = f(0) + f(1). \tag{9}$$

Hint: you should probably define $C = f(0) + f(1)$

Hidden constant function

Find all functions $f(x)$ such that

$$f'(x)f(-x) = 1, \qquad f(0) = 4. \tag{10}$$

Hint: you should look at $g(x) = f(x)f(-x)$