Directions: This examination lasts 4 hours.

- Partial credit will be awarded to relevant work.
- No credit will be awarded for unexplained answers, correct or not.
- Computations mistakes will be very lightly penalized.
- Accurate graphic representation of a problem will receive high consideration.
- All questions are worth the same number of points.

1) Suppose A is diagonalizable and can be factored as $A = S\Lambda S^{-1}$ where the columns of S are the eigenvectors and Λ is the diagonal matrix of eigenvalues. Find the eigenvalues and eigenvectors of $A + 2I$ in terms of S and Λ.

2) Explain why a diagonalizable matrix is not necessarily invertible.

3) Show that any upper triangular matrix T that is unitary must be diagonal.

4) The singular value decomposition of a square matrix A is given as $A = U\Sigma V^H$ with unitary matrices U and V, and a diagonal matrix $\Sigma = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$. Show that the 2-norm of A defined as

$$\|A\|_2 = \max_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2},$$

is given by the maximum diagonal entry of Σ, *i.e.*

$$\|A\|_2 = \max_{1\leq i\leq n} \sigma_i.$$

5) Determine and sketch the location of all unit-norm vectors in $\mathbb{R}^2$ where the norm is taken as the (a) 1-norm, (b) 2-norm and (c) ∞-norm.

6) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is positive definite, test A^{-1} for positive definiteness.

7) Show that the vectors of the following basis

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

are linearly independent and construct an equivalent orthogonal basis.

8) Let M be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$. Prove that M^2 is a linear transformation also.

9) Find dimensions and bases for the four fundamental subspaces: column space, nullspace, row space and left nullspace for

$$A = \begin{bmatrix} 0 & 3 & 3 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}.$$

10) A and B are diagonalizable matrices and satisfy $AB = BA$. Show that A and B have the same eigenvectors. Do they have the same eigenvalues?