Duration: 4 Hours
Instructions: Show your work, credit will not be given to answers without explanation. Partial credit will be awarded for correct work, unless otherwise specified. When you are asked to explain yourself, please write clearly and use complete sentences. Good luck!

When you are asked to *explain* your reasoning, you must use complete sentences.

1. True or False: Provide a short explanation for each case. (Remember a true statement must ALWAYS be true.)
   (a) Let $\vec{b} \in \mathbb{R}^4$ and define $S = \{\vec{y} \in \mathbb{R}^4 | \vec{y}^T \vec{b} = 0\}$. S is a subspace of $\mathbb{R}^4$.
   (b) If A is an $n \times n$ diagonalizable matrix, then 0 can not be an eigenvalue of A.
   (c) Let A, B and C be $n \times n$ matrices. If A is similar to B and B is similar to C, then A must be similar to C.

2. Consider the following linear system $A\vec{x} = \vec{b}$ where,

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{bmatrix}$$

   and $\vec{b} = \begin{bmatrix} b \\ b \\ b \end{bmatrix}$ for a real number b. Determine the values of b where so that $A\vec{x} = \vec{b}$ has at least 1 solution and solve the linear system.

3. Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$, compute A^{105}.

4. Let S be the subspace formed by e^x, xe^x and x^2e^x. Let D be the differentiation operator on S.
   (a) Prove that D is a linear transformation.
   (b) Find the matrix representing D with respect to $[e^x, xe^x, x^2e^x]$.

5. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

   (a) Find the singular value decomposition (SVD).
   (b) From the SVD indicate the corresponding orthonormal bases for each: $C(A), N(A), C(A^T)$ and $N(A^T)$.

6. Let U be a 3-dimensional subspace of $\mathbb{R}^6$ with basis vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and P be the matrix which projects $\mathbb{R}^6$ onto U.
   (a) What are the eigenvalues of P?
   (b) What are the eigenvectors of P.
   (c) Let $\vec{x} \in \mathbb{R}^6$ be an arbitrary vector, describe the limit as $n \to \infty$ of $\vec{x}_n = P^n \vec{x}$.
7. If A is square matrix:
   (a) Show that the nullspace of A^2 contains the nullspace of A.
   (b) Show that the column space of A^2 is contained in the column space of A.
8. Let Q be an $n \times n$ orthogonal matrix, show that $\|Q\| = 1$. Note that by $\|.\|$ for a matrix mean the usual matrix norm where:

$$\|A\| = \max_{\vec{x} \neq \vec{0}} \frac{\|Ax\|}{\|x\|}$$

   and $\|x\|$ is the usual Euclidean norm of a vector in $\mathbb{R}^n$.
9. Let A be an $n \times n$ symmetric matrix. Show that if $\lambda_1 \neq \lambda_2$ are eigenvalues of A then their corresponding eigenvectors $\vec{x}_1$ and $\vec{x}_2$ must be orthogonal.