Duration: 4 Hours

Instructions: Show your work, credit will not be given to answers without explanation. Partial credit will be awarded for correct work, unless otherwise specified. When you are asked to explain yourself, please write clearly and use complete sentences. Good luck!

When you are asked to *explain* your reasoning, you must use complete sentences.

Notation: In the questions below, if A is a matrix, we use the following notation:

$C(A)$ = Column space of A.

$N(A)$ = Null space of A.

1. State whether each of the following are True or False and provide a short justification (few sentences) to justify your decision.
   (a) If A is a symmetric positive definite matrix, then A is invertible.
   (b) If A and B are $n \times n$ matrices with the same characteristic polynomial they are similar.
   (c) If $A\vec{x} = \vec{0}$ has infinitely many solutions, then there exists a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ does not have any solution for an $m \times n$ matrix A where $n < m$.
   (d) If A has the same 4 fundamental subspaces $(C(A), N(A), C(A^T), N(A^T))$ as a matrix B, then $A = B$.

2. Let L be the linear operator defined on P_3, the vector space of polynomials of maximal degree 2 defined by:
$$L\big(p(x)\big) = xp'(x) + p''(x).$$
   (a) Prove that L is a linear transformation.
   (b) Find the matrix A representing L with respect to the basis $\{1, x, x^2\}$.
   (c) Find all eigenvectors of this operator, polynomials of maximal degree 2 where: $L\big(p(x)\big) = \alpha p(x)$ where $\alpha \in \mathbb{R}$.
   (d) If $p(x) = a_0 + a_1 x + a_2 x^2$, calculate:
$$L^n(p(x)).$$

3. If A is a 2×2 symmetric matrix with eigenvectors $\vec{v}_1$ and $\vec{v}_2$ with corresponding eigenvalues λ_1 and λ_2. Write the singular value decomposition (SVD) for A.

4. Let $\vec{x}$ and $\vec{y}$ be as defined below:
$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}. \tag{1}$$
   (a) Find the matrix P that projects vectors in $\mathbb{R}^4$ onto $\vec{y}$.

(b) Without explicitly calculating them, determine the eigenvalues of P and their respective multiplicities.

(c) Find the projection $\vec{p}$ of $\vec{x}$ onto $\vec{y}$.

(d) The Pythagorean Law holds between $\vec{p}, \vec{x}$ and one other vector. Find this other vector and verify the Pythagorean Law holds.

5. Let N be a square matrix where $N \neq 0$ but $N^k = 0$ where 0 is the zero matrix.

(a) Find all possible eigenvalues, λ that satisfy:

$$N\vec{x} = \lambda\vec{x}.$$

(b) Can N be diagonalizable? Prove or give a counter example.

(c) Can N be symmetric? Prove or give a counter example.

6. Consider the following matrix A:

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix}$$

(a) Determine the QR factorization of the matrix A.

(b) Use this factorization to determine the least squares solution of:

$$A\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}.$$

7. Let A be a symmetric $n \times n$ matrix. Prove the following statements hold:

(a) $\|A\|_\infty = \|A\|_1$
where $\|.\|_1$ and $\|.\|_\infty$ are the matrix norm induced by the standard vector norms for $\vec{x} \in \mathbb{R}^n$:

$$\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i| \text{ and}$$

$$\|\vec{x}\|_\infty = \max_{i=1,\dots n} |x_i|.$$

Recall that a matrix norm, $\|.\|_M$, is induced by a vector norm, $\|.\|_V$ if the following holds:

$$\|A\|_M = \max_{\|\vec{x}\|_V=1} \|A\vec{x}\|_V.$$

(b) $\|A\|_2 = \rho(A)$
where $\rho(A)$ is the spectral radius of the matrix, the largest absolute value $|\lambda|$ of an eigenvalue of A.