Applied Mathematics ODE Preliminary Exam January 2015

Instructions. Solve each of the problems below. Credit will awarded mainly based on the quality of your work and the clarity of your explanations.

1. Solve the inital value problem,

$$y' = (1+y^2)\tan x, \quad y(0) = 1.$$

2. Find the general solution of

$$2y'' + 3y' + y = x^2 + 3\sin x.$$

3. For the differential equation,

$$\dot{x} = e^{-x} - \cos x,$$

sketch the vector field on the real line, find all the fixed points, and classify their stability. It is sufficient to find the approximate locations of the fixed points.

Use your results to find $\lim_{t\to\infty} x(t)$, when $x(0) = -\pi/2$.

4. Newton's law of cooling states that the temperature, $T(t)$, of an object in surroundings of temperature, $A(t)$, is governed by the differential equation,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - A),$$

with $k > 0$ measuring the rate that heat is absorbed or emitted by the object. Now consider the following scenario.

> *I make a cup of coffee at 7:00 am using boiling water. After adding some milk, I measure the temperature of the coffee and find it to be* $90°C$*. I get distracted by answering emails and forget about the coffee. When I come back to the coffee at 7:30 am, I measure its temperature and find it to be* $45°C$*. At that exact moment, I receive a phone call that I need to answer. After that phone call, I finally get back to the coffee at 8:00 am and measure its temperature to be* $30°C$*.*

Using Newton's law of cooling, find the temperature of my house. Identify any assumptions that you make in determining this temperature.

5. Sketch the phase portrait for the following system

$$\begin{aligned} \dot{x} &= x(1 - x + y), \\ \dot{y} &= y(1 + x - 2y), \end{aligned}$$

for $x, y \geq 0$.

This system models the populations of two, cooperating species. Interpret your phase portrait in this context.

6. Show that the differential equation

$$xy'' + (1-x)y' + \lambda y = 0,$$

possesses polynomial solutions if λ is a nonnegative integer. Show that these polynomial solutions exist by seeking a power series solution of this differential equation about $x = 0$ and finding that these series terminate. Evaluate these polynomial solutions for $\lambda \leq 4$, standardizing them by the condition $y(0) = 1$.

7. Solve the differential equation,

$$x^2 y'' + x^2 y' - 2y = 0,$$

about $x = 0$ using the method of Frobenius. Show that one of the series terminates. Express the second solution in closed form (using a method of your choice).

8. Solve the following eigenvalue problem

$$x^2 y'' - \lambda x y' + \lambda y = 0, \quad \text{in } 1 < x < 2$$
$$y(1) = y(2) = 0.$$

Find all of the eigenvalues and eigenfunctions. Hint: Use the guess $y(x) = x^r$.

Is it a *Sturm-Liouville* eigenvalue problem? Explain why or why not using your solution.