Where are the surface plasmons?

Asymptotics for the cavity case

Zoïs Moitier

Waves Seminar, UC Merced

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UNIVERSITY OF CALIFORNIA MERCED

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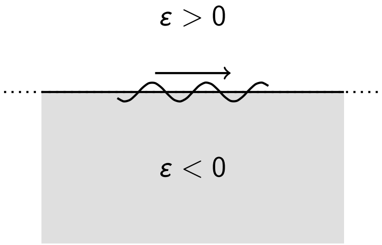
Carvalho Lab for exOtic waVes in Electromagnetics, Resonances and Scattering

Link with Ben's talk



For $\beta \in \mathbb{R}^*$, there exists a wavenumber $k \in \mathbb{C}$ such that

$$u(x, y) = w(y)\, e^{i\beta x}$$

is a surface plasmon.

For $m \in \mathbb{Z}^*$, there exists a wavenumber $k \in \mathbb{C}$ such that

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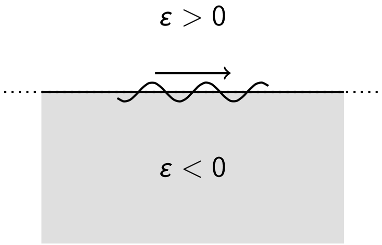

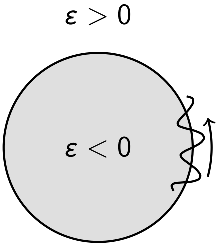

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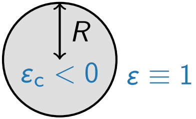
Permittivity ε, piecewise constant, discontinuous across C_R

- $\varepsilon \equiv \varepsilon_c < 0$ in the cavity D_R;
- $\varepsilon \equiv 1$ in $\mathbb{R}^2 \setminus \overline{D_R}$.

D_R (disk) and C_R (circle) of radius R.



Resonances problem: Find $(\ell^2, u) \in \mathbb{C} \times \mathrm{H}^1_{\mathrm{loc}}\left(\mathbb{R}^2\right)$, $u \not\equiv 0$, such that

$$\begin{cases} -\operatorname{div}(\varepsilon^{-1} \nabla u) = \ell^2 u & \text{in } D_R \text{ and } \mathbb{R}^2 \setminus \overline{D_R} \\ [u]_{C_R} = 0 \quad \text{and} \quad \left[\varepsilon^{-1} \partial_{\mathbf{n}} u\right]_{C_R} = 0 & \text{across } C_R \\ u \text{ is } \ell\text{-outgoing} & \end{cases}$$

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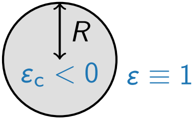

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u is ℓ-outgoing means that for $r > R$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$

$$u(r,\theta) = \sum_{m\in\mathbb{Z}} \Big(a_m\, \mathrm{H}^{(1)}_m(\ell r) + \underbrace{b_m}_{=0}\, \mathrm{H}^{(2)}_m(\ell r) \Big) \mathrm{e}^{\mathrm{i}m\theta}.$$

The spectral problem

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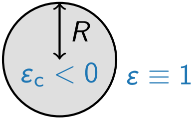

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- The operator $-\mathrm{div}(\varepsilon^{-1} \nabla)$ is self-adjoint if, and only if, $\varepsilon_c \neq -1$ [Costabel & Stephan, 1985].
- It is not semibounded, $\mathrm{sp}_{\mathrm{dis}} \subset \mathbb{R}^*_-$ unbounded and $\mathrm{sp}_{\mathrm{ess}} = \mathbb{R}_+$ [Carvalho & Moitier, in preparation].

Almost-explicit computation for circular cavities

- Fourier series expansion in the periodic angular variable θ:

$$u(r,\theta) = \sum_{m\in\mathbb{Z}} w_m(r)\, \mathrm{e}^{\mathrm{i}m\theta}.$$

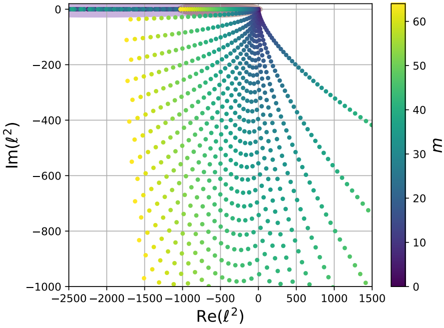
- $\ell^2 \in \mathbb{C}$ is a solution of the spectral problem if, and only if, there exists $m \in \mathbb{Z}$ such that

$$\mathrm{I}'_m(\eta\,\ell R)\,\mathrm{H}^{(1)}_m(\ell R) + \eta\,\mathrm{I}_m(\eta\,\ell R)\mathrm{H}^{(1)'}_m(\ell R) = 0$$

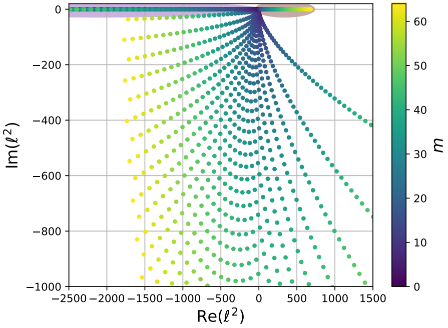
where $\eta = \sqrt{-\varepsilon_c} > 0$.

- The associated resonant mode

$$u_\ell(r,\theta) = \mathrm{e}^{\mathrm{i}m\theta} \begin{cases} \mathrm{I}_m(\eta\,\ell\,r) & \text{if } r \le R \\ \frac{\mathrm{I}_m(\eta\,\ell\,R)}{\mathrm{H}^{(1)}_m(\ell\,R)}\,\mathrm{H}^{(1)}_m(\ell\,r) & \text{if } r > R \end{cases}.$$



$\varepsilon_c = -0.8$

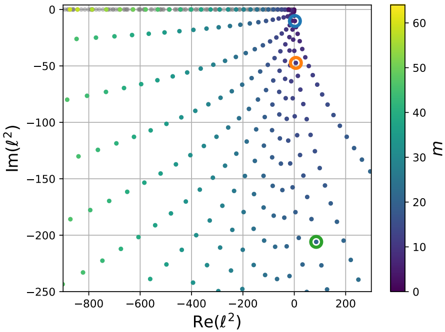


$\varepsilon_c = -1.2$

For $R = 1$ and $\eta = \sqrt{-\varepsilon_c}$, graphs in $(\mathrm{Re}(\ell^2), \mathrm{Im}(\ell^2))$ of the roots of $\ell \mapsto \mathrm{I}'_m(\eta\,\ell R)\,\mathrm{H}^{(1)}_m(\ell R) + \eta\,\mathrm{I}_m(\eta\,\ell R)\mathrm{H}^{(1)\,\prime}_m(\ell R)$.

This set can be partitioned $\mathcal{R}[\eta] = \mathcal{R}_{\mathrm{out}}[\eta] \cup \mathcal{R}_{\mathrm{inn}}[\eta] \cup \mathcal{R}_{\mathrm{pla}}[\eta]$.

Outer resonances for a circular cavity



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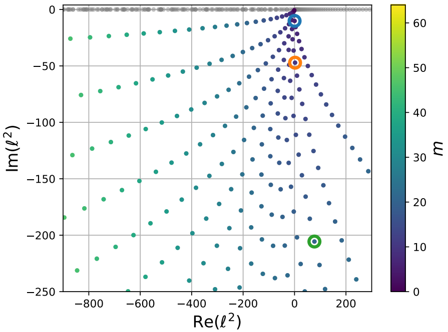

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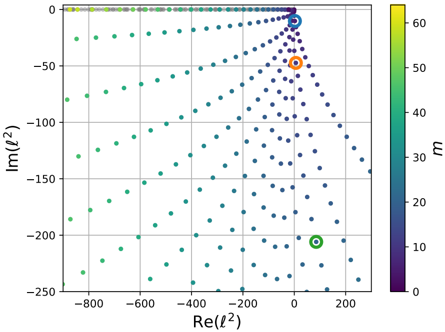
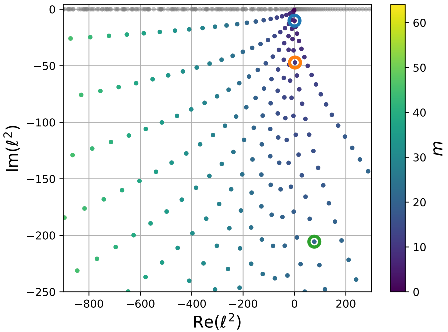
The outer resonances "live" outside of the cavity.





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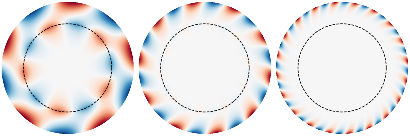
$m = 5$ $m = 10$ $m = 20$



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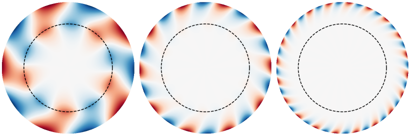
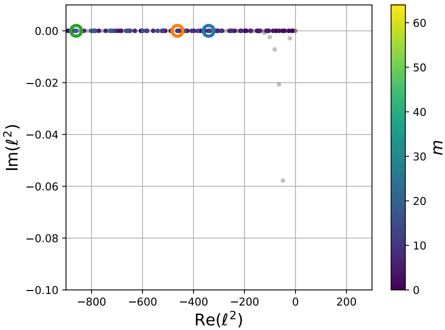


Inner resonances for a circular cavity



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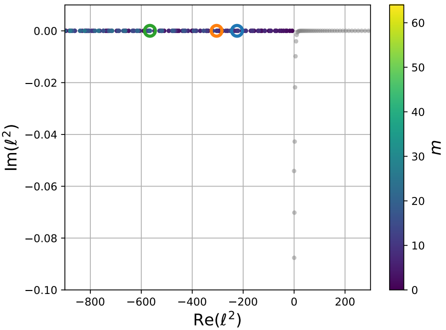

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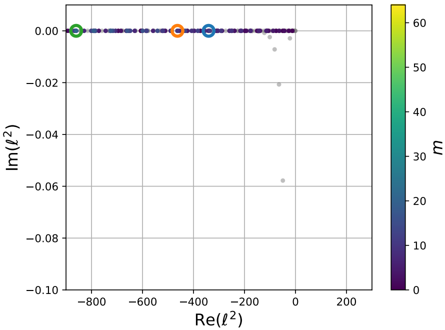
The inner resonances are negative eigenvalues ($\ell^2 < 0$) of $-\mathrm{div}(\varepsilon^{-1}\nabla)$ on $\mathrm{L}^2(\mathbb{R}^2)$ and “live” inside the cavity.

Inner resonances for a circular cavity



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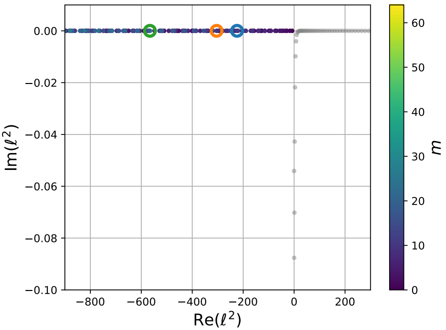

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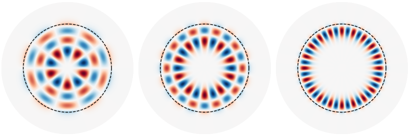

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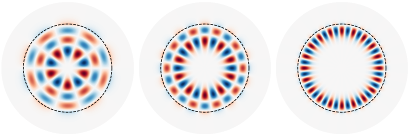
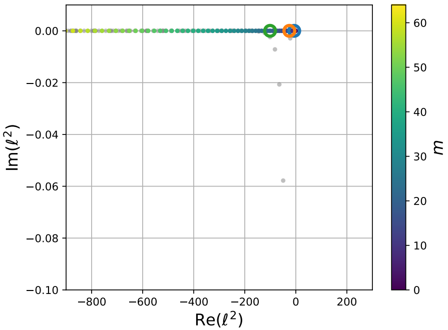


Plasmonic resonances for a circular cavity



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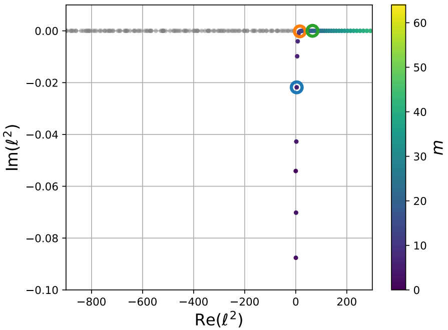
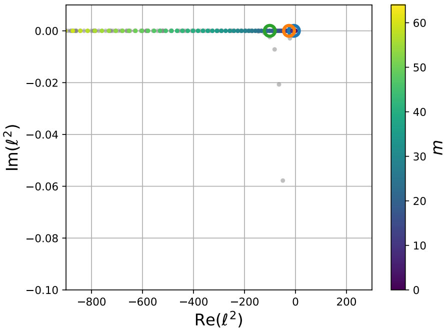


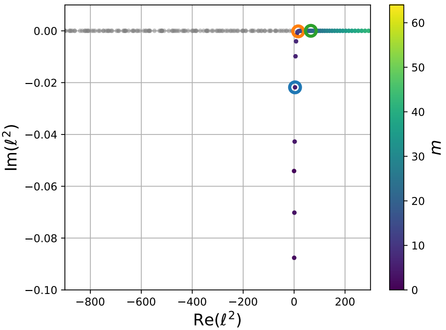
$\varepsilon_c = -1.2$

The plasmonic resonances "live" on the interface and

- for $-1 < \varepsilon_c < 0$, are negative eigenvalues ($\ell^2 < 0$);
- for $\varepsilon_c < -1$, are resonances ($\mathrm{Re}(\ell^2) > 0$ and $-1 \ll \mathrm{Im}(\ell^2) < 0$).

Plasmonic resonances for a circular cavity





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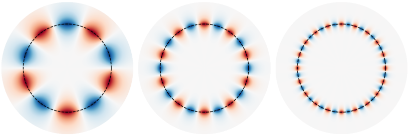

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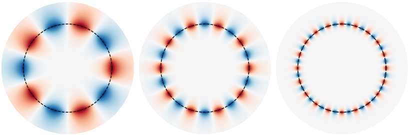
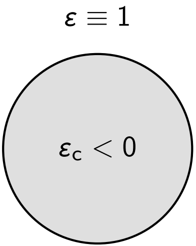
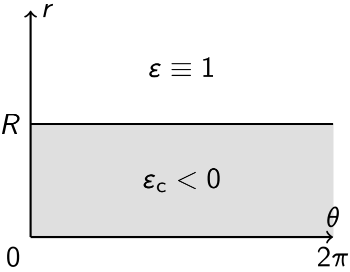




Cartesian coordinates: $(x, y) \in \mathbb{R}^2$

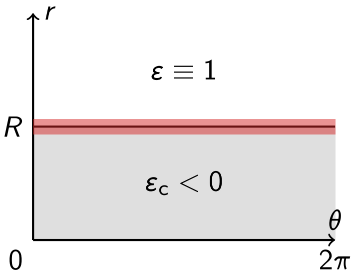
$$\begin{cases} -\mathrm{div}\left(\varepsilon^{-1}\nabla u\right) = \ell^2 u \\ [u]_{C_R} = 0 \\ \left[\varepsilon^{-1}\partial_{\mathbf{n}} u\right]_{C_R} = 0 \end{cases}$$

Scaling of the plasmon as $m \to +\infty$



Polar coordinates: $(r, \theta) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$

$$\begin{cases} -\frac{1}{r}\partial_r \left(\varepsilon^{-1}\, r\partial_r u\right) - \frac{\varepsilon^{-1}}{r^2}\partial_\theta^2 u = \ell^2 u \\ [u]_{\{R\}\times\mathbb{R}/2\pi\mathbb{Z}} = 0 \\ \left[\varepsilon^{-1}\,\partial_r u\right]_{\{R\}\times\mathbb{R}/2\pi\mathbb{Z}} = 0 \end{cases}$$



Polar coordinates: $(r, \theta) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$

$$u(r,\theta) = w(r)\,\mathrm{e}^{\mathrm{i}m\theta} \quad \text{with } m \in \mathbb{Z}^*$$

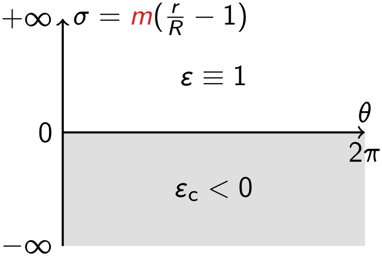
$$\begin{cases} -\frac{1}{r}\partial_r\left(\varepsilon^{-1}\, r\partial_r w\right) - m^2\frac{\varepsilon^{-1}}{r^2}w = \ell^2 w \\ [w]_{\{R\}} = 0 \\ \left[\varepsilon^{-1}\,\partial_r w\right]_{\{R\}} = 0 \end{cases}$$

Scaling of the plasmon as $m \to +\infty$

Scale coordinates: $(\sigma, \theta) \in \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$

$$\varphi(\sigma) = w(r) \quad \text{and} \quad \underline{\lambda} = \left(\frac{R\ell}{m}\right)^2$$

$$\begin{cases} -\partial_\sigma \left(\varepsilon^{-1} \partial_\sigma \varphi\right) - \varepsilon^{-1}\varphi = \underline{\lambda}\varphi + \mathcal{O}(\frac{1}{m}) \\ [\varphi]_{\{0\}} = 0 \\ \left[\varepsilon^{-1} \partial_\sigma \varphi\right]_{\{0\}} = 0 \end{cases}$$



Theorem: For $\varepsilon_c \neq -1$, there exists a sequence $(\ell_m)_{m\geq 1}$ of complex such that

- $(\ell_m^2)_{m\geq 1}$ are negative eigenvalues for $-1 < \varepsilon_c < 0$,
- $(\ell_m^2)_{m\geq 1}$ are resonances for $\varepsilon_c < -1$,

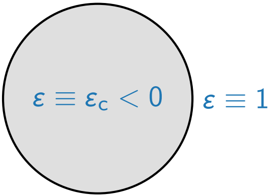
and

$$\ell_m^2 = \frac{m^2}{R^2}\left(1+\varepsilon_c^{-1}\right)\left[1+\sum_{q=1}^{N-1}\lambda_q m^{-q} + \mathcal{O}\left(m^{-N}\right)\right], \quad \forall N \geq 1.$$

Remark: All the coefficients λ_q are real and, for $\varepsilon_c < -1$, we have $0 < 1+\varepsilon_c^{-1} < 1$ so $\operatorname{Im}(\ell_m^2) = \mathcal{O}(m^{-N})$ for all $N \in \mathbb{N}$.

Proof: It relies on the *Black Box Scattering* theory and the theorem of [Tang & Zworski, 1998].

The scattering problem



Given:

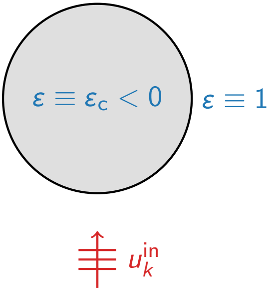
- a wavenumber $k > 0$
- an incident field $u_k^{\mathrm{in}}(x, y) = e^{i k y}$

Find: the scattered field $u_k^{\mathrm{sc}} \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that $u = u_k^{\mathrm{in}} + u_k^{\mathrm{sc}}$ and

$$\begin{cases} -\mathrm{div}(\varepsilon^{-1} \nabla u) - k^2 u = 0 & \text{in } \mathbb{R}^2 \setminus C_R \\ [u]_{C_R} = 0 & \text{across } C_R \\ \left[\varepsilon^{-1} \partial_{\mathbf{n}} u\right]_{C_R} = 0 & \text{across } C_R \\ u_k^{\mathrm{sc}} \text{ is } k\text{-outgoing} & \end{cases}$$

- u_k^{sc} is k-outgoing $\Leftrightarrow$ Sommerfeld radiation condition.
- The problem is well posed for $\varepsilon_c \neq -1$ ($\mathbb{T}$-coercivity arguments).

The scattering problem



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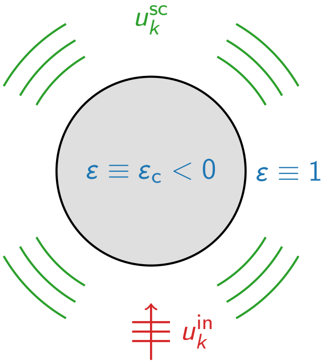

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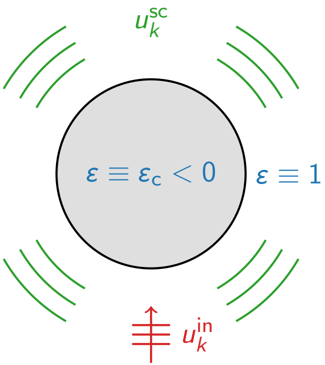

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Almost-explicit computation for circular cavities

The incident field with the Jacobi-Anger expansion:

$$u_k^{\text{in}}(x,y) = \mathrm{e}^{\mathrm{i}\,k\,y} = \mathrm{e}^{\mathrm{i}\,k\,r\sin(\theta)} = \sum_{m\in\mathbb{Z}} \mathrm{J}_m(k\,r)\,\mathrm{e}^{\mathrm{i}m\theta}.$$

The scattered field:

$$u_k^{\text{sc}}(x,y) = \sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}m\theta} \begin{cases} a_m\,\mathrm{I}_m(k\,r) - \mathrm{J}_m(k\,r) & \text{if } r \le R \\ b_m\,\mathrm{H}_m^{(1)}(k\,r) & \text{if } r > R \end{cases}$$

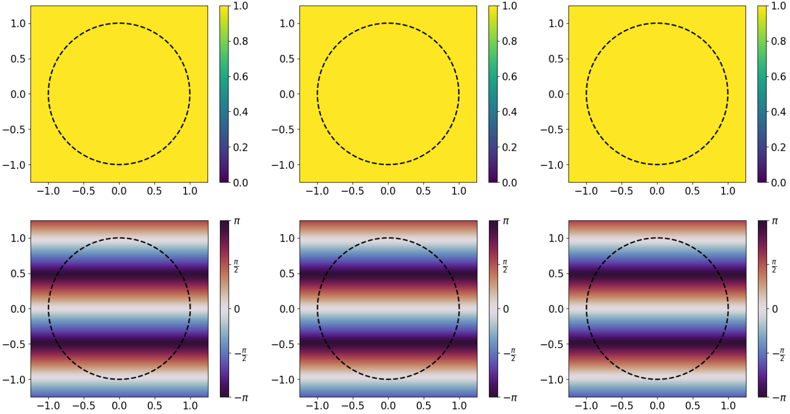
where, with $\eta = \sqrt{-\varepsilon_{\mathrm{c}}}$, the coefficients (a_m, b_m) solve

$$\begin{pmatrix} \mathrm{I}_m(\eta kR) & -\mathrm{H}_m^{(1)}(kR) \\ \eta^{-1}\,\mathrm{I}'_m(\eta kR) & \mathrm{H}_m^{(1)\prime}(kR) \end{pmatrix} \begin{pmatrix} a_m \\ b_m \end{pmatrix} = \begin{pmatrix} \mathrm{J}_m(kR) \\ -\mathrm{J}'_m(kR) \end{pmatrix}.$$

For $R = 1$ and $\varepsilon_c = -1.2$, the graphs of $u_k^{\text{in}}(x, y) = e^{iky}$.

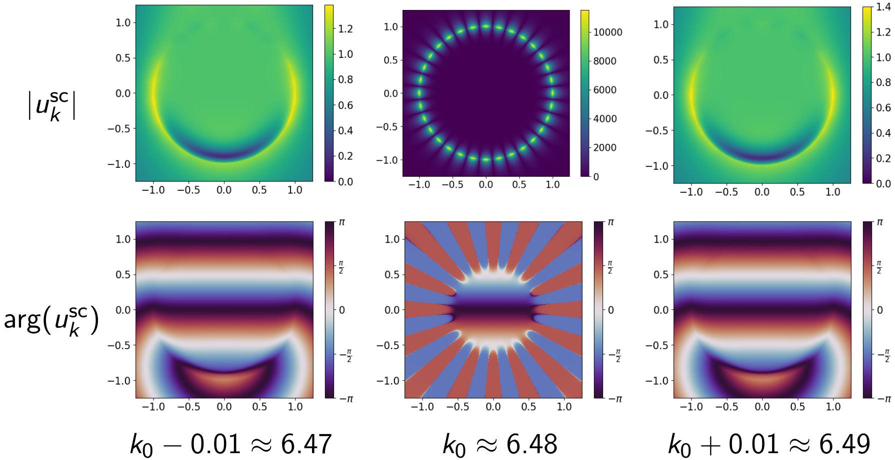
$|u_k^{\text{in}}|$

$\arg(u_k^{\text{in}})$

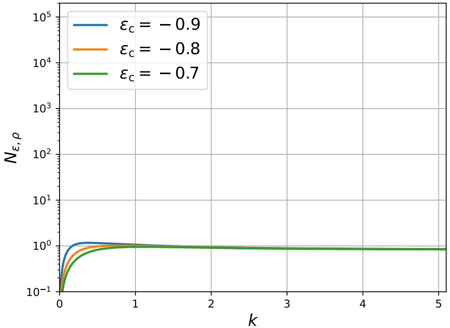


$k_0 - 0.01 \approx 6.47$ $\quad$ $k_0 \approx 6.48$ $\quad$ $k_0 + 0.01 \approx 6.49$

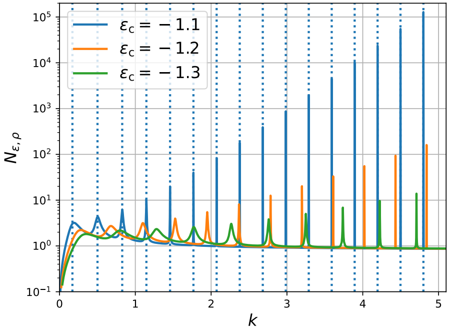
For $R = 1$ and $\varepsilon_c = -1.2$, the graphs of u_k^{sc}.



$R = 1$, $\rho = 2$, and graphs of $N_{\varepsilon,\rho} : k \mapsto \frac{\|u_k^{\mathrm{sc}}\|_{L^2_\rho}}{\|u_k^{\mathrm{in}}\|_{L^2_\rho}}$ where $L^2_\rho = L^2(D(0,\rho))$



$\varepsilon_c \in \{-0.9, -0.8, -0.7\}$



$\varepsilon_c \in \{-1.1, -1.2, -1.3\}$

The blue dashed lines correspond to $\mathrm{Re}(\ell_m)$ of the plasmonic resonances for $\varepsilon_c = -1.1$.

Final remarks

Remark:

- Everything in this talk is valid for other shape of cavity and/or variable permittivity.

Conclusions:

- We can excite surface plasmons via scattering only if $\varepsilon_c < -1$ because they correspond to resonances close to $\mathbb{R}_+$.
- But this is bad news for FEM and BEM because those surface plasmons constrained the meshes with their high number of oscillations and localization along the interface of the cavity. Can we extract them?

Thank you for your attention

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Thank you for your attention