Asymptotic approximation of Boundary Integral Equations with high curvature

Lori Lewis (Master Student)
Advisors : Dr. C. Carvalho and Dr. A. D. Kim

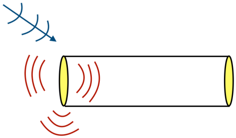
University of California, Merced

Outline

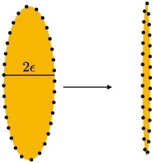
Outline

Motivation

- Slender Body Theory [1]
  - How light is affected when obstacles have a region of high curvature.



- Wave problem : Exterior Helmholtz
- Focus on 2D in Laplace

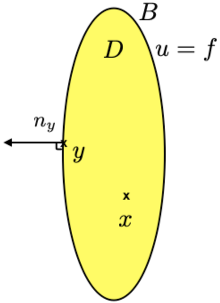


Outline

Introduction

Interior Dirichlet Laplace problem:

$$\Delta u = 0 \quad \text{in} \quad D$$
$$u = f \quad \text{on} \quad B$$



The solution is represented as a double-layer potential

$$u(x) = \int_B \frac{\partial G}{\partial n_y}(x,y)\mu(y)\mathrm{d}\sigma_y, \quad x \in D$$

Green's function $G(x,y) = -\frac{1}{2\pi}\log|x-y|$

μ is continuous density that satisfies the BIE

$$-\frac{1}{2}\mu(x) + \int_B \frac{\partial G}{\partial n_y}(x,y)\mu(y)\mathrm{d}\sigma_y = f(x), \quad x \in B$$

where $\frac{\partial G}{\partial n_y}(x,y) = -\frac{1}{2\pi} n_y \cdot \frac{x-y}{|x-y|^2}$

Why do we use Boundary Integral method ?

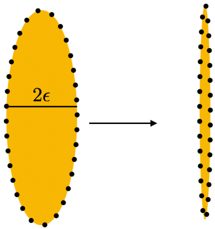
+ Reduction of 1 dimension
+ High-order methods available [2]
– Dense matrices

Challenge:

- We focus on the boundary integral equation

$$-\frac{1}{2}\mu(x) + \int_B \frac{\partial G}{\partial n_y}(x,y)\mu(y)\mathrm{d}\sigma_y = f(x), \quad x \in B$$

  for a region of high curvature.
- We have a parameter ε which perturbs our ellipse to have the following behavior



Solving the BIE

To solve the boundary integral equation we take advantage of parameterization

$$y(s) = \langle \varepsilon \cos s, \sin s \rangle, \quad 0 \le s \le 2\pi$$

Plugging it into the boundary integral equation we get

$$-\frac{1}{2}\mu(s) + \frac{1}{2\pi}\int_0^{2\pi} n_y(t) \cdot \frac{y(s)-y(t)}{|y(s)-y(t)|^2}|y'(t)|\mu(t)\mathrm{d}t = f(s), \quad 0 \le s \le 2\pi$$

The boundary integral equation can be rewritten as

$$-\frac{1}{2}\mu(s) + \frac{1}{2\pi}\int_0^{2\pi} K(s,t;\varepsilon)\mu(s)\mathrm{d}t = f(s), \quad 0 \le s \le 2\pi$$

with kernel

$$K(s,t;\varepsilon) = \frac{\varepsilon}{-1-\varepsilon^2-(1-\varepsilon^2)\cos(s+t)}$$

Taking a closer look at the kernel

$$K(s,t;\varepsilon) = \frac{\varepsilon}{-1-\varepsilon^2-(1-\varepsilon^2)\cos(s+t)}$$

Notice that K is nearly singular [3] when $s+t=\pi$ or $s+t=3\pi$ which is a challenge.

When $\cos(s+t) = -1$ then

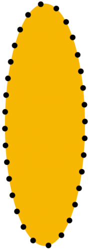
$$K(s,t;\varepsilon) = -\frac{1}{2\varepsilon}$$

$$\lim_{\varepsilon\to 0} K(s,t;\varepsilon) \to -\infty$$

Numerical Investigation: Periodic Trapezoid Rule

Recall the BIE:

$$-\frac{1}{2}\mu(s)+\frac{1}{2\pi}\int_0^{2\pi}K(s,t;\varepsilon)\mu(t)\mathrm{d}t=f(s),\quad 0\leq s\leq 2\pi$$



The **Periodic Trapezoid Rule**

$$-\frac{1}{2}\mu(s_i)+\frac{1}{N}\sum_{j=1}^{N}K(s_i,t_j)\mu(t_j)=f(s_i),\quad i=1,\dots,N$$

$$\text{where}\quad s_i=\frac{2\pi i}{N}\quad\text{and}\quad t_j=\frac{2\pi j}{N}$$

We solve the **System** $(-\frac{1}{2}I_N+P)\mu_N=f_N$
where P is the matrix obtained from the Periodic Trapezoid rule.

Recall the BIE :

$$-\frac{1}{2}\mu(s)+\frac{1}{2\pi}\int_0^{2\pi}K(s,t;\varepsilon)\mu(s)\mathrm{d}t=f(s)$$

Gauss' Law[3]

$$\int_B \frac{\partial G}{\partial n_y}(x,y)\mu(y)\mathrm{d}\sigma_y=\begin{cases}-1, & x\in D\\ -\frac{1}{2}, & x\in B\\ 0, & x\in E\end{cases}$$

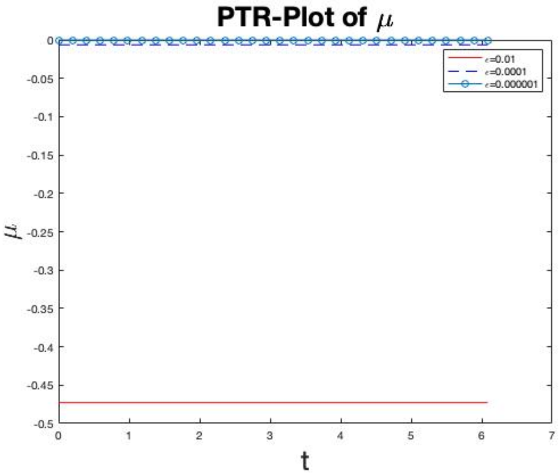
Example: We run a text case where $f=1$. Since f is constant and we assume μ to be constant then Gauss' Law will give us the following

$$-\frac{1}{2}\mu(s)-\frac{1}{2}\mu(s)=1$$

$$\mu(s)=-1$$

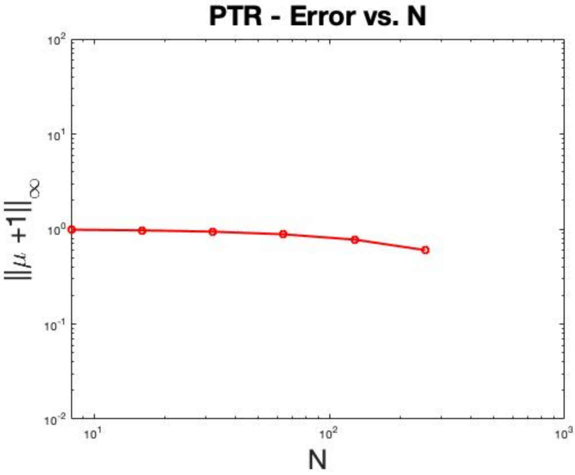
The **test case solution** to the BIE will be $\mu=-1$.

We run PTR for $N = 32$ quadrature points and different values of ε.



- PTR approximation is way off $\mu(s) \neq -1$, which is a **problem**.

The following is a log-log plot of the error vs. N for the periodic trapezoid rule with a fixed $\varepsilon = 0.001$

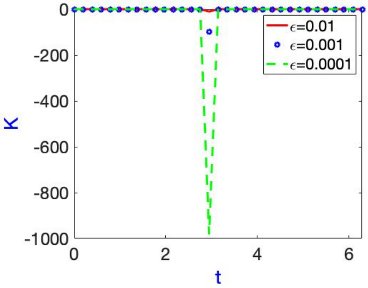


Reason:

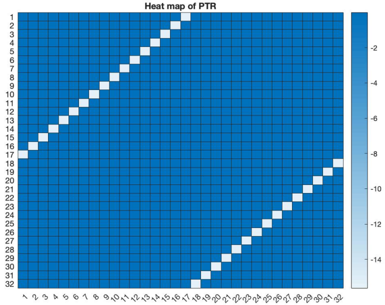
Recall, $K(s,t;\varepsilon) = -\frac{1}{2\varepsilon}$ when $\cos(s+t) = -1$

then

$$K(s,t;\varepsilon) = -\frac{1}{2\varepsilon}, \quad \lim_{\varepsilon \to 0} K(s,t;\varepsilon) \to -\infty$$



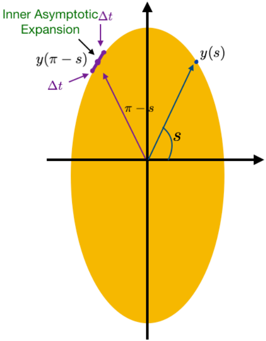
(a) Plot of kernel for different values of ε.



(b) Heat map of matrix P for PTR.

Inner Asymptotic expansion

We provide an alternative numerical method for the points $y(s+t) = y(\pi)$ and $y(s+t) = y(3\pi)$.



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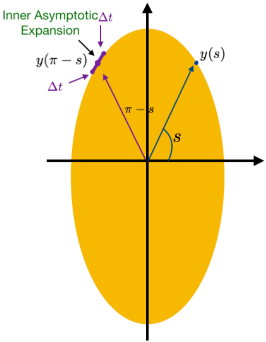


- The expression of the integral for a neighborhood around the point $y(\pi - s)$

$$I_1 = \int_{\pi-s-\Delta t}^{\pi-s+\Delta t} K(s,t;\varepsilon)\mu(t)\mathrm{d}t$$

Substitutions:

1. $t = \tau + \pi - s$ with

$$\mathrm{d}t = \mathrm{d}\tau \quad \longrightarrow \quad I_1 = \frac{1}{2\pi}\int_{-\Delta t}^{\Delta t} K(s, \tau + \pi - s; \varepsilon)\mu(\tau + \pi - s)\mathrm{d}\tau$$

2. $\tau = \varepsilon T$ with

$$\mathrm{d}\tau = \varepsilon \mathrm{d}T \quad \longrightarrow \quad I_1 = \frac{1}{2\pi}\int_{-\frac{\Delta t}{\varepsilon}}^{\frac{\Delta t}{\varepsilon}} K(s, \varepsilon T + \pi - s; \varepsilon)\mu(\varepsilon T + \pi - s)\varepsilon \mathrm{d}T$$

Expanding about $\varepsilon = 0$:

$$K(s, \varepsilon T + \pi - s; \varepsilon) = \frac{1}{(-2 - \frac{T^2}{2})\varepsilon} + \frac{(-12T^2 - T^4)\varepsilon}{6(4 + T^2)^2} + O(\varepsilon^3)$$

$$\mu(\varepsilon T + \pi - s) = \mu(\pi - s) + \varepsilon T \mu(\pi - s) + O(\varepsilon^2)$$

$$I_1 \sim \frac{1}{2\pi}\int_{-\frac{\Delta t}{\varepsilon}}^{\frac{\Delta t}{\varepsilon}} \frac{\mu(\pi - s)}{\left(-2 - \frac{T^2}{2}\right)\varepsilon}\varepsilon \mathrm{d}T + O(\varepsilon) \sim -\frac{\arctan(\frac{\Delta t}{2\varepsilon})\mu(\pi - s)}{\pi}$$

Modified numerical method:

At the points $y(s+t) = y(\pi)$ and $y(s+t) = y(3\pi)$ we replace PTR with the asymptotic.

$$-\frac{1}{2}\mu(s_i) + \frac{1}{N}\sum_{j=1}^{N} K(s_i,t_j)\mu(t_j) = f(s_i), \quad i = 1, \ldots .N$$

Modified BIE

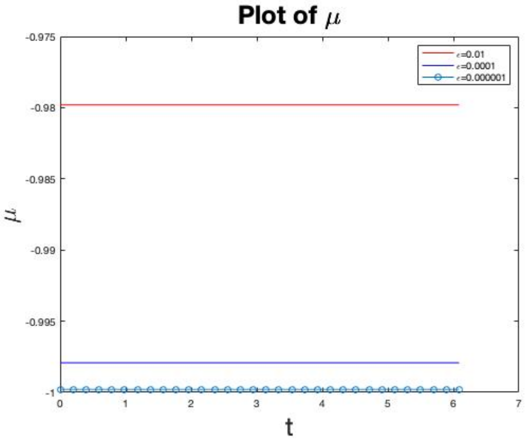
$$-\frac{1}{2}\mu(s_i) + \frac{1}{N}\sum_{\substack{j=1 \\ s_i+t_j\neq\pi \\ s_i+t_j\neq 3\pi}}^{N} K(s_i,t_j)\mu(t_j) + \sum_{\substack{s_i+t_j=\pi \\ s_i+t_j=3\pi}} \left(-\frac{\arctan(\frac{\Delta t}{2\varepsilon})\mu(\pi-s)}{\pi}\right) = f(s_i),$$

We solve the **System** $i = 1, \ldots .N$

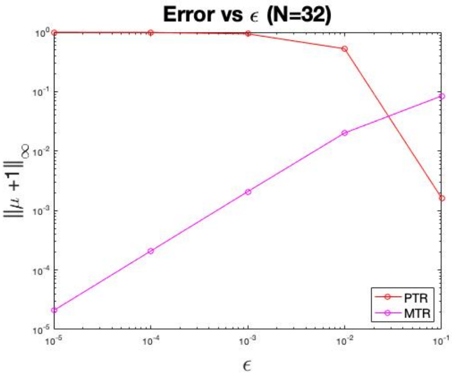
$$(-\frac{1}{2}I_N + P_m)\mu_N = f_N$$

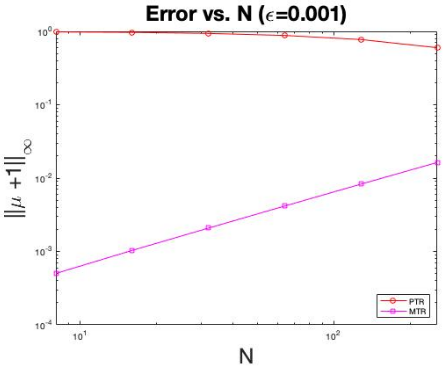
where P_m is the Matrix of the MPTR.

Numerical Results: we run the MTR for $N = 32, f = 1$ and different values of ε (test case $\mu = -1$).



Comparing the Error





Modified trapezoid rule

- The Method does not do well when $\Delta t < \varepsilon$.

Trapezoid rule

- Requires more points to get a good approximation of μ

Spectral approximation of the BIE :

Fourier Series

Recall that the boundary integral equation

$$-\frac{1}{2}\mu(s) + \frac{1}{2\pi}\int_0^{2\pi} K(s,t;\varepsilon)\mu(t)\mathrm{d}t = f(s), \quad 0 \leq s \leq 2\pi$$

Steps:

1. Substitute

$$\mu(s) = \sum_{n=-\infty}^{\infty} \hat{\mu}_n e^{ins}, \;\; K(s,t;\varepsilon) = \sum_{m=-\infty}^{\infty} \hat{k}_m e^{im(s+t)}, \;\; \text{and} \;\; f(s) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{ins}$$

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The **Fourier representation of the BIE** is given by

$$-\frac{1}{2}\hat{\mu}_n+\hat{\mu}_{-n}\hat{k}_n=\hat{f}_n, \qquad n=-\infty,\ldots,\infty$$

Finding the Fourier coefficients

1. We rewrite $K(s,t;\varepsilon)$ as a **rational trigonometric function** [4]. $\rightarrow$ $K(s,t;\varepsilon) = \left(\frac{-\varepsilon}{1+\varepsilon^2}\right)\left(\frac{1}{1+\frac{1-\varepsilon^2}{1+\varepsilon^2}\cos(s+t)}\right)$

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2. Let $c_0 = \frac{-\varepsilon}{1+\varepsilon^2}$ and $c_1 = \frac{1-\varepsilon^2}{1+\varepsilon^2}$ with $c_1 < 1$ $\rightarrow$ $K(s,t;\varepsilon) = \left(\frac{c_0}{1+c_1\cos(s+t)}\right)$

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3. We obtain → $\hat{k}_n = c_0 \frac{1+\rho^2}{1-\rho^2}\rho^{|n|}, \quad \forall n.$

   with $I_{n,1} = \rho^n \frac{1+\rho^2}{1-\rho^2} \quad n \geq 0$

   and $\rho = \frac{\sqrt{1-c_1^2}-1}{c_1}$

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   with $I_{n,1} = \rho^n \frac{1+\rho^2}{1-\rho^2} \quad n \geq 0$

   and $\rho = \frac{\sqrt{1-c_1^2}-1}{c_1}$

The **truncated system** becomes

$$-\frac{1}{2}\hat{\mu}_n + \hat{\mu}_{-n}\hat{k}_n = \hat{f}_n, \qquad n = -\frac{N}{2},\ldots,\frac{N}{2}-1 \tag{1}$$

FFT Results

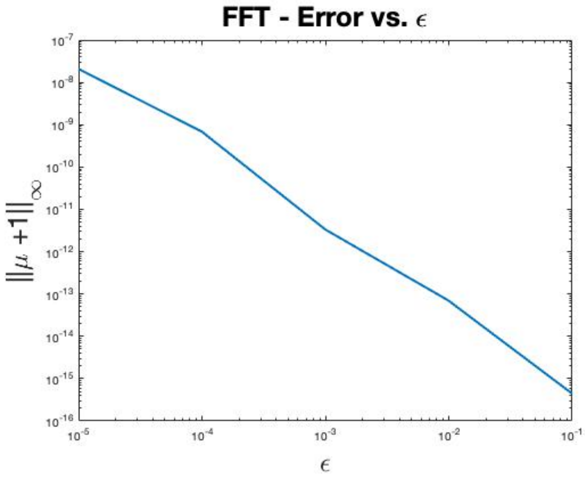

FIGURE 3 – Plot of the error VS. ε of Spectral method for different values of epsilon.

Summary - Interior Dirchlet Laplace

	PTR	MTR	FFT
Approximation	-	+	+
Accuracy	-	+	++
Limitations	+	+	-

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Extension of the methods to

- Exterior Neumann Laplace

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	PTR	MTR	FFT
Approximation	-	+	+
Accuracy	-	+	++
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Extension of the methods to

- Exterior Neumann Laplace
- Scattering Problem

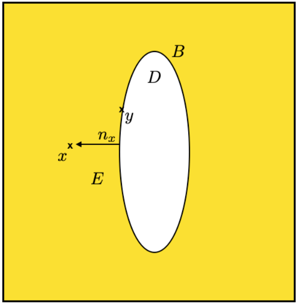
Outline

Exterior Neumann

Exterior Neumann Laplace problem:

$$\Delta u = 0 \quad \text{in} \quad E$$
$$\frac{\partial u}{\partial n_x} = f \quad \text{on} \quad B$$

where $E := \mathbb{R}^2/\bar{D}$



The solution is represented as a single-layer potential

$$u(x) = \int_B G(x,y)\mu(y)\mathrm{d}\sigma_y, \quad x \in E$$

Green's function $\quad G(x,y) = -\frac{1}{2\pi}\log|x-y|$

μ is continuous density that satisfies the BIE

$$\frac{1}{2}\mu(x) - \int_B \frac{\partial G}{\partial n_x}(x,y)\mu(y)\mathrm{d}\sigma_y = f(x), \quad x \in B$$

Nearly Singular Behavior

using parameterization $y(s) = \langle \varepsilon \cos s, \sin s \rangle, \quad 0 \leq s \leq 2\pi$ We get the BIE writren as

$$\frac{1}{2}\mu(s) + \frac{1}{2\pi}\int_0^{2\pi} K(s,t;\varepsilon)\mu(s)\mathrm{d}t = f(s), \quad 0 \leq s \leq 2\pi \tag{1}$$

with kernel

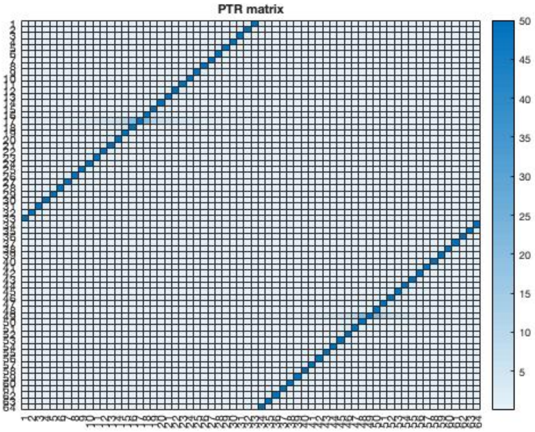
$$K(s,t;\varepsilon) = -\frac{\varepsilon\sqrt{1+\varepsilon^2-(-1+\varepsilon^2)\cos(2t)}}{\sqrt{2}(-1-\varepsilon^2+(-1+\varepsilon^2)\cos(s+t))\sqrt{\cos^2(s)+\varepsilon^2\sin^2(s)}}$$

Notice that K is nearly singular when $s+t=\pi$ or $s+t=3\pi$. when $\cos(s+t)=-1$ then

$$K(s,\pi-s;\varepsilon) = -\frac{\sqrt{1+\varepsilon^2-(-1+\varepsilon^2)\cos(2(\pi-s))}}{2\sqrt{2}\varepsilon\sqrt{\cos^2(s)+\varepsilon^2\sin^2(s)}}$$

$$\lim_{\varepsilon\to 0} K(s,t;\varepsilon) \to -\infty$$

PTR system : $(\frac{1}{2}I_N + P_N)\mu_N = f_N$.

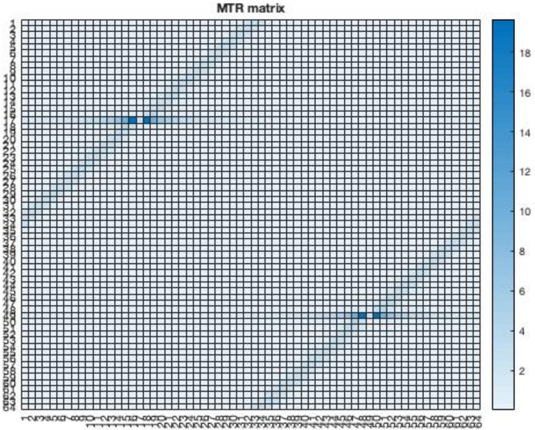


Similar to Interior Dirichlet Laplace problem :

- Inner Expansion
- Spectral Method

Inner Asymptotic Expansion :

$$I_1 = \int_{\pi-s-\Delta t}^{\pi-s+\Delta t} K(s,t;\varepsilon)\mu(s)\mathrm{d}t \sim 2\arctan(\frac{\Delta t}{2\varepsilon}) + O(\varepsilon)$$



Challenges :

- No exact Solution
- No analytic Fourier Series Coefficients available

Spectral Method

- Do direct FFT - Requires a lot of quadrature points

Scattering Problem

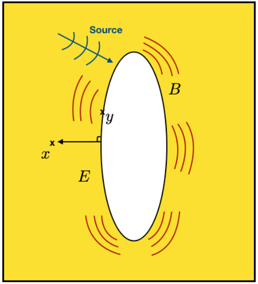
Exterior Helmholtz problem:

$$\begin{aligned} \Delta u + k^2 u = 0, &\quad \text{in} \quad E \\ u = f, &\quad \text{on} \quad B. \end{aligned}$$

$+$ radiation condition

$\downarrow$

The solution is represented as a double- and single-layer potentials



$$u(x) = \int_B \left(\frac{\partial G}{\partial n_y}(x,y) - ikG(x,y) \right) \mu(y) \mathrm{d}\sigma_y, \quad x \in E$$

Green's function $G(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|)$

μ is continuous density that satisfies

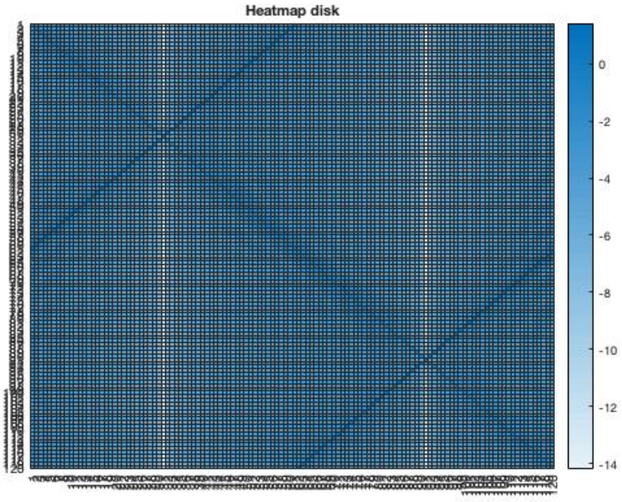
$$\frac{1}{2}\mu(x)+\int_B\left(\frac{\partial G}{\partial n_y}(x,y)-ikG(x,y)\right)\mu(y)\mathrm{d}\sigma_y=f(x),\quad x\in B$$

Challenges

- Singular kernel at $x=y$
- ε affects the kernel

We use Kress quadrature [5] with N quadrature points to solve the **system**

$$(\frac{1}{2}I_N+P_H)\mu_N=f_N$$



Current Investigation : Inner Asymptotic Expansion

Outline

Conclusion

- Regions of high curvature affect BIE
- Periodic Trapezoid Rule not always effective.
- To address this we do an asymptotic expansion and sometimes use Spectral method.
- Work in progress :
  - Helmholtz
  - Extension to other boundary shapes

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Thank you !