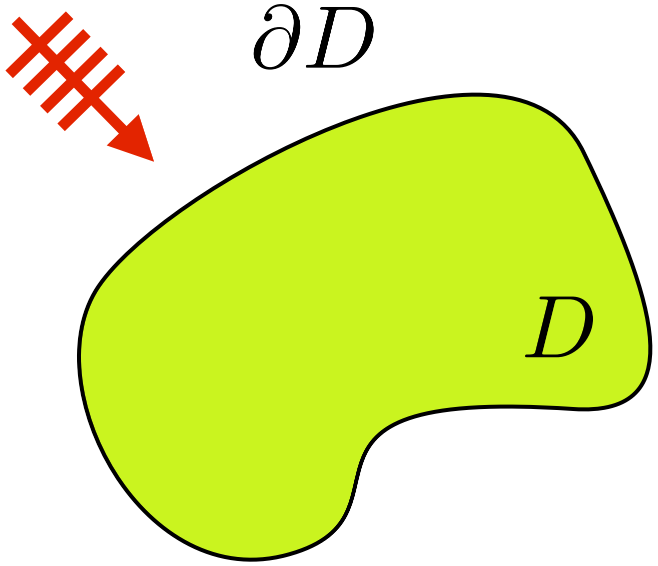
Subtraction Techniques for the close evaluation of layer potentials

Camille Carvalho

WAVES Seminar, 2020

Introduction: 2D close evaluation problem



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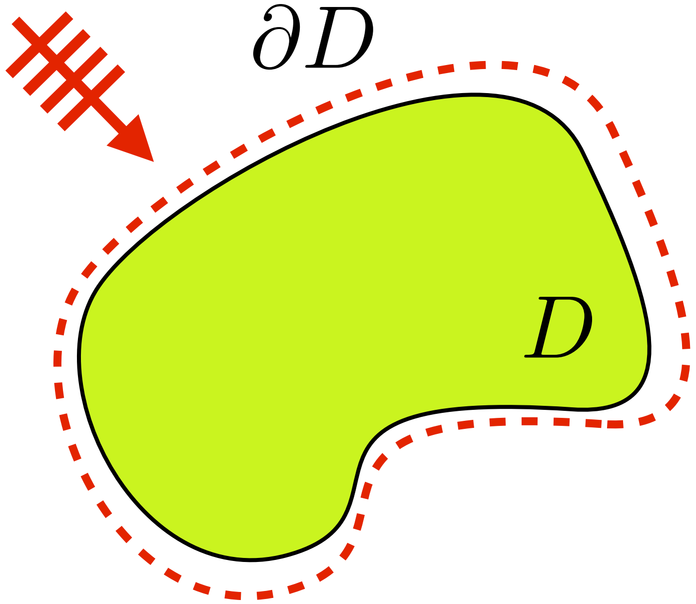

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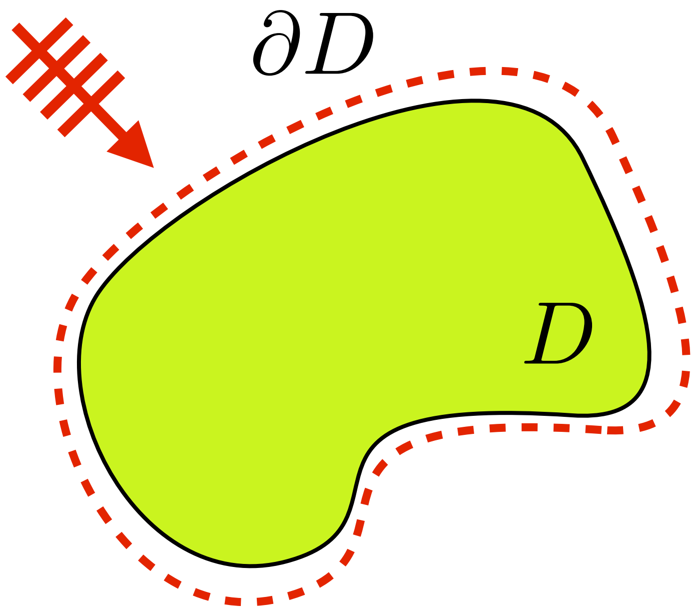

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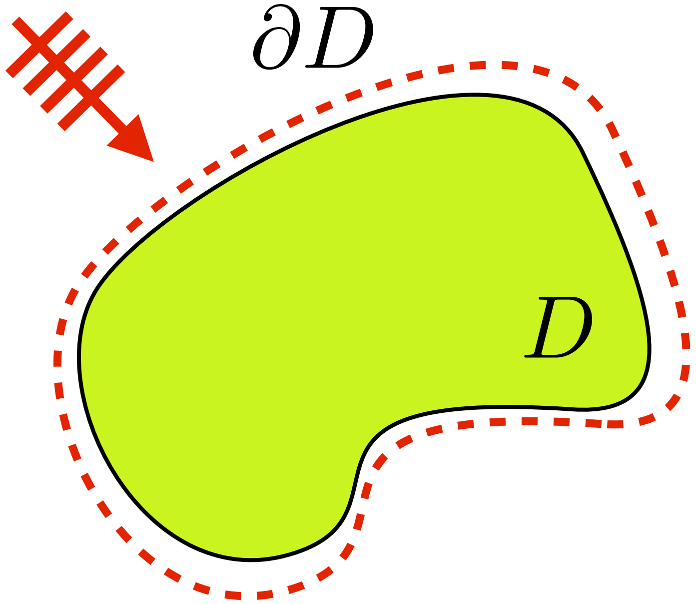

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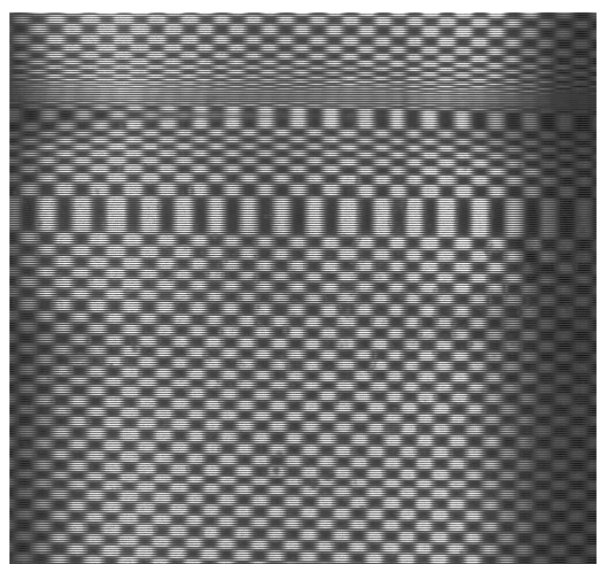

JFL Lab.

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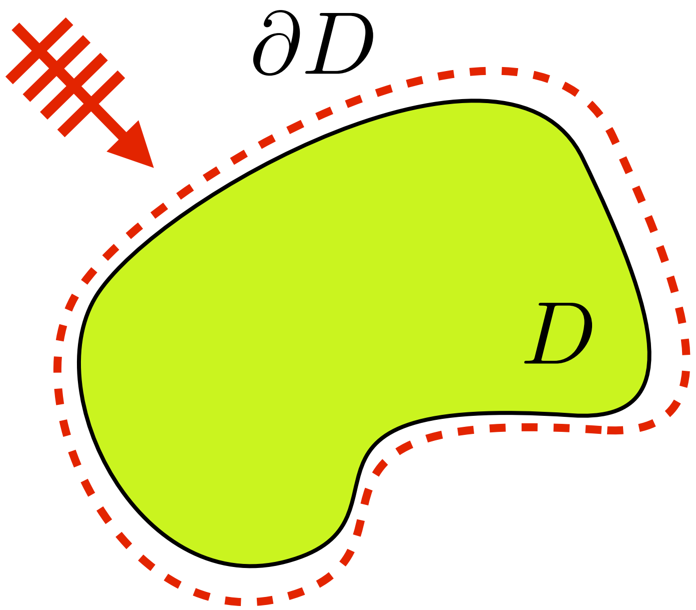

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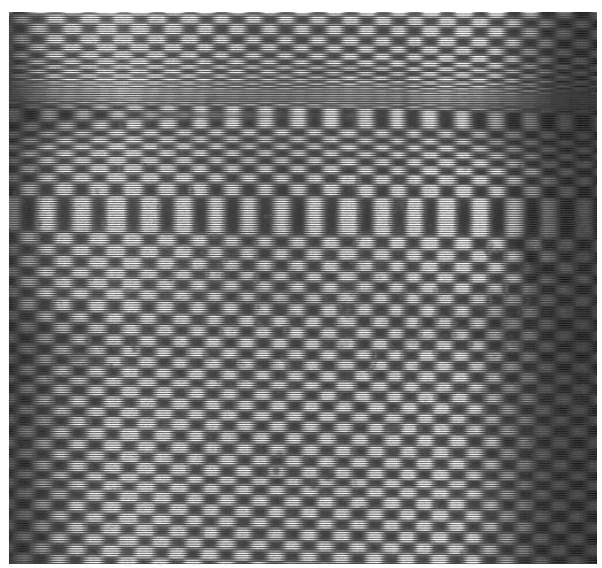

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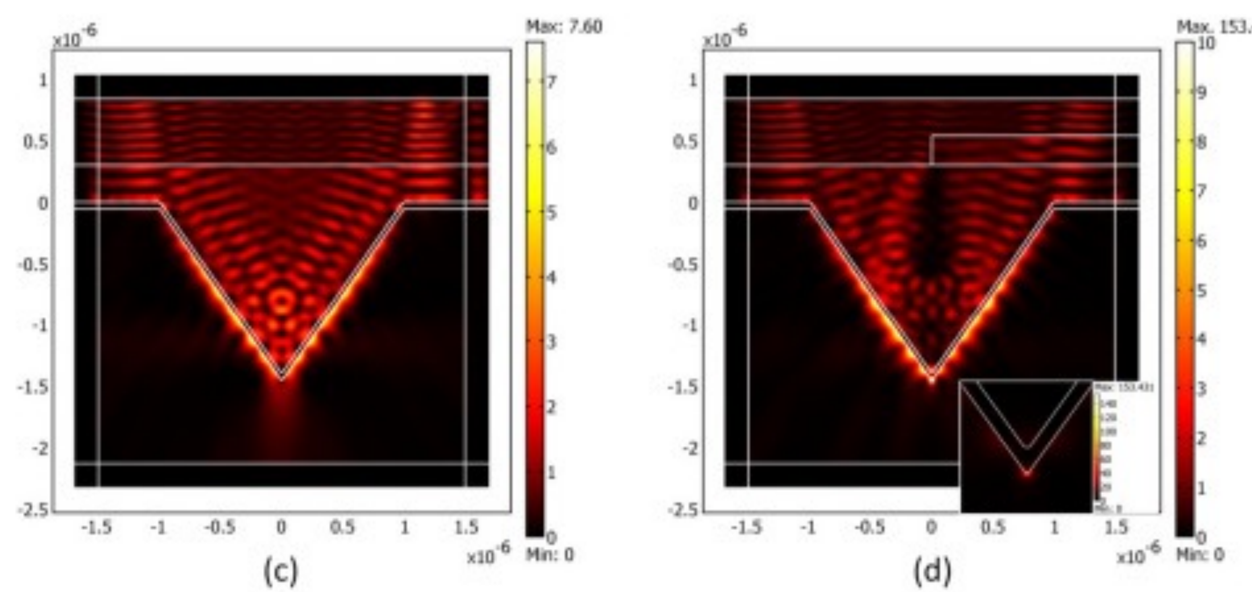

Garoli et al., (2015)

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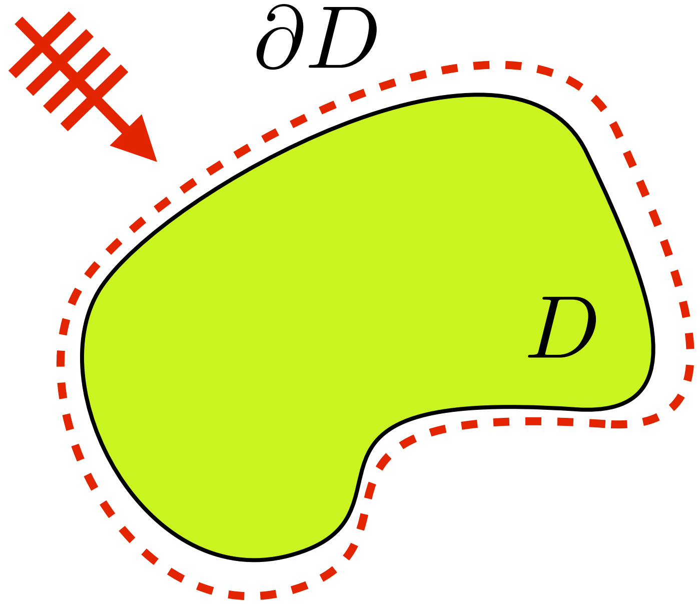

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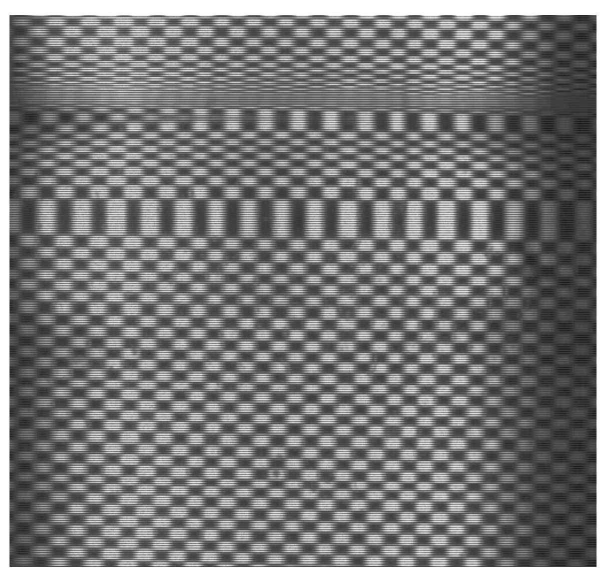

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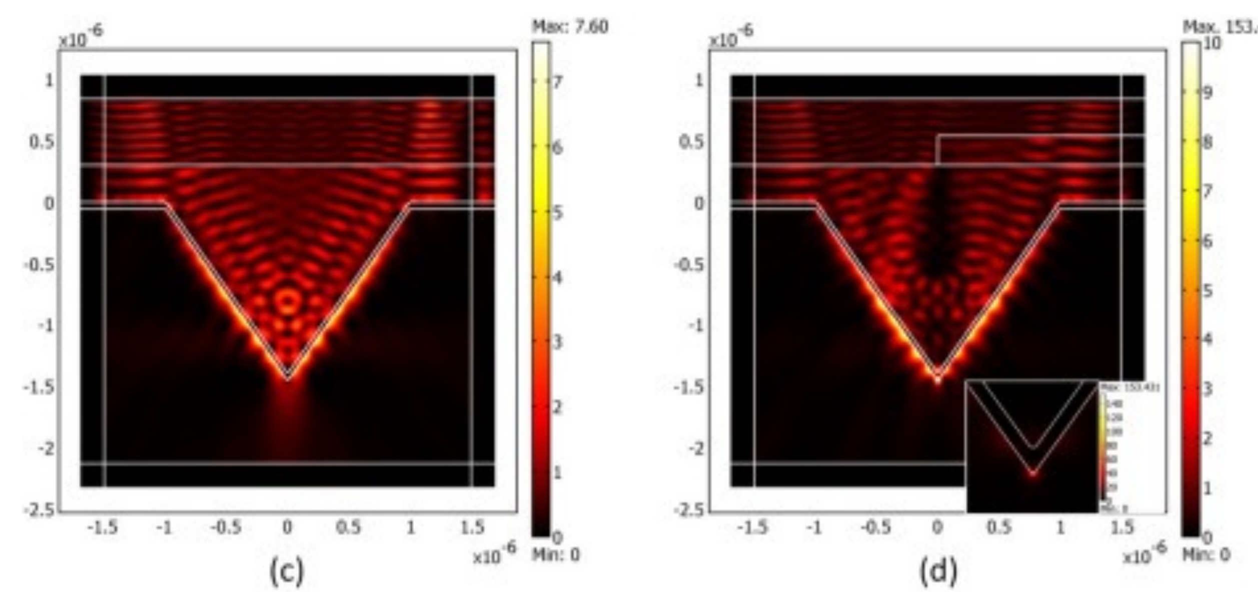

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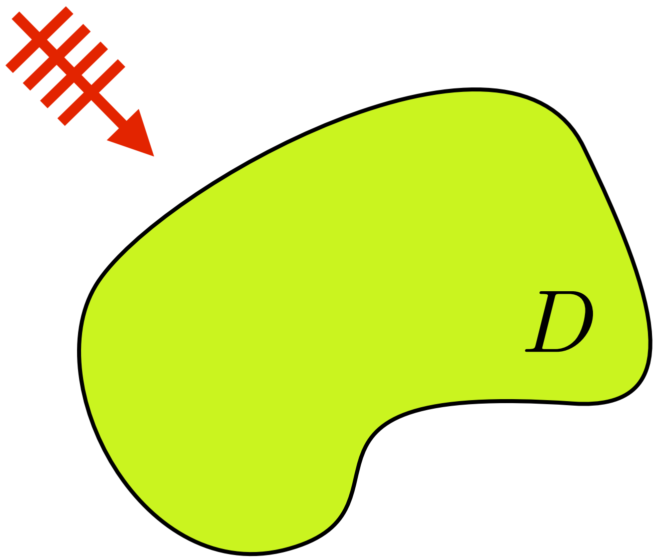
How ? Using boundary integral methods.

Boundary integral methods



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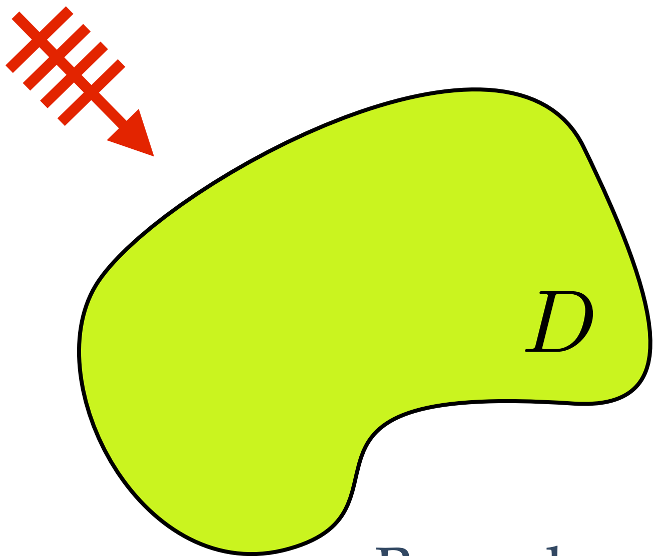

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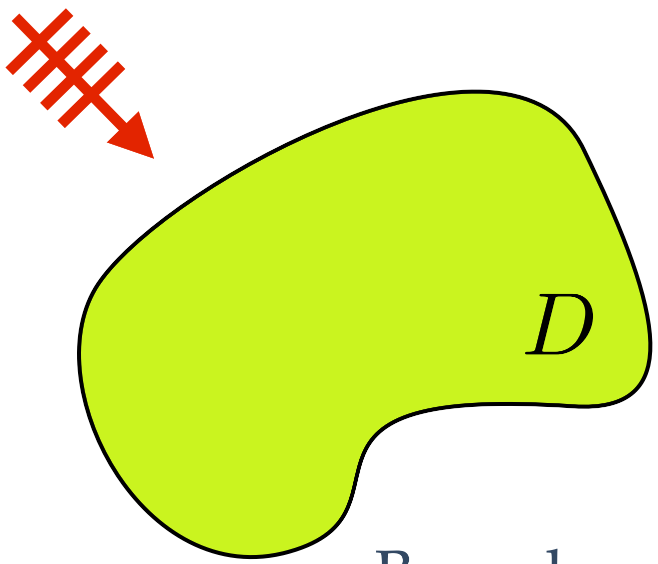

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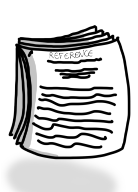

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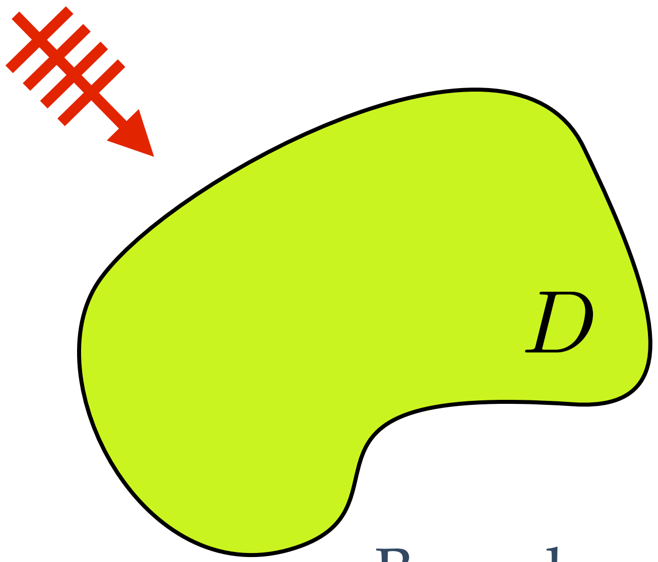

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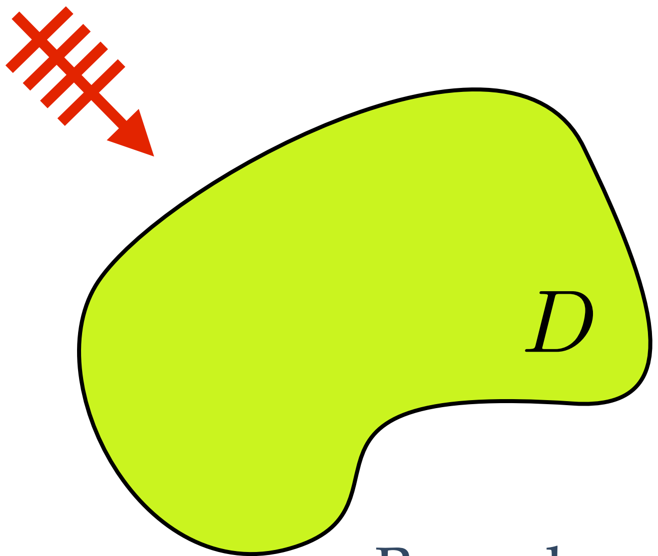

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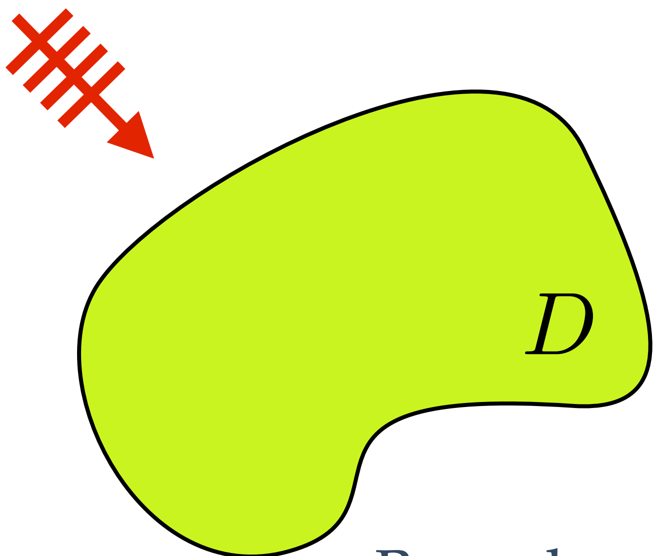

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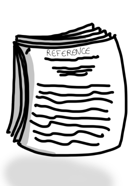
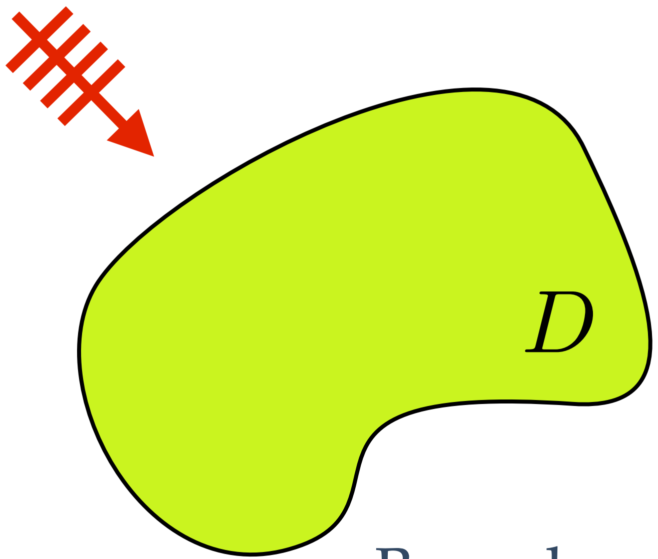


Kress (1991).

✔ reduces the problem by one dimension

Boundary integral methods



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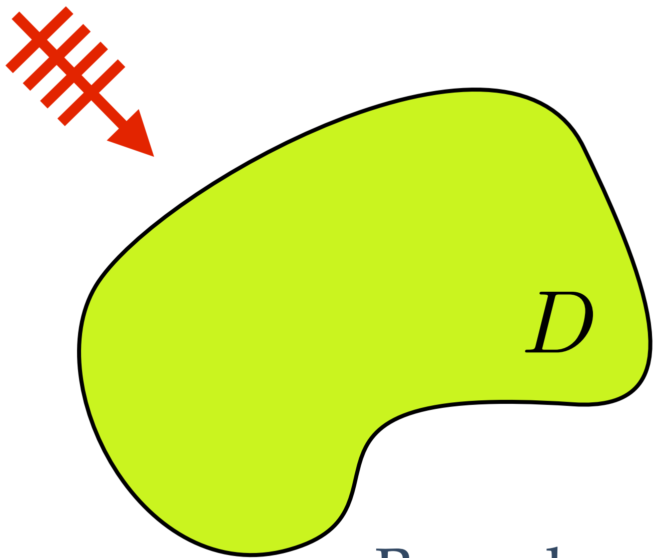

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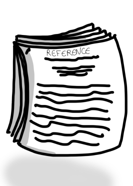

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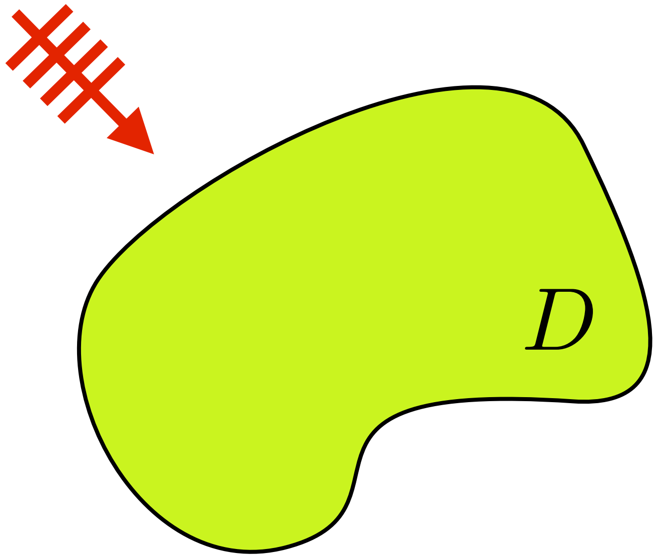

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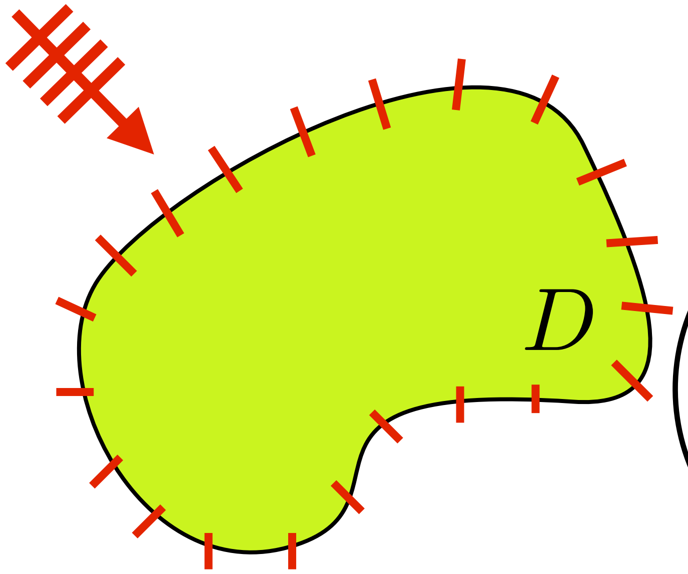

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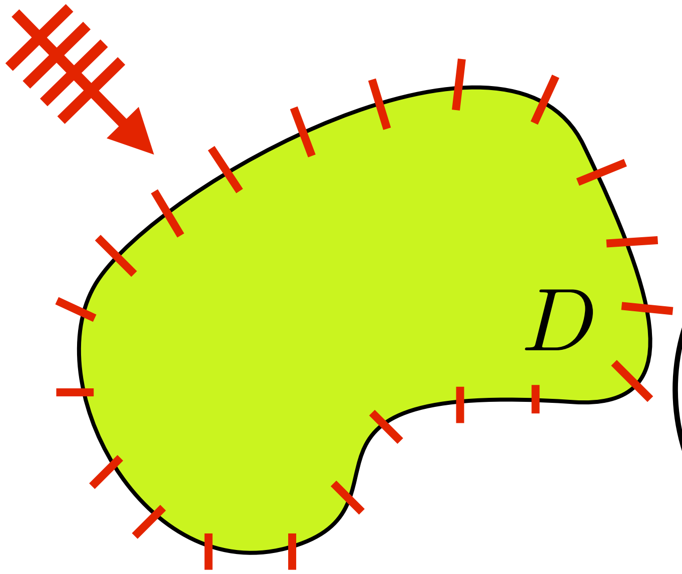

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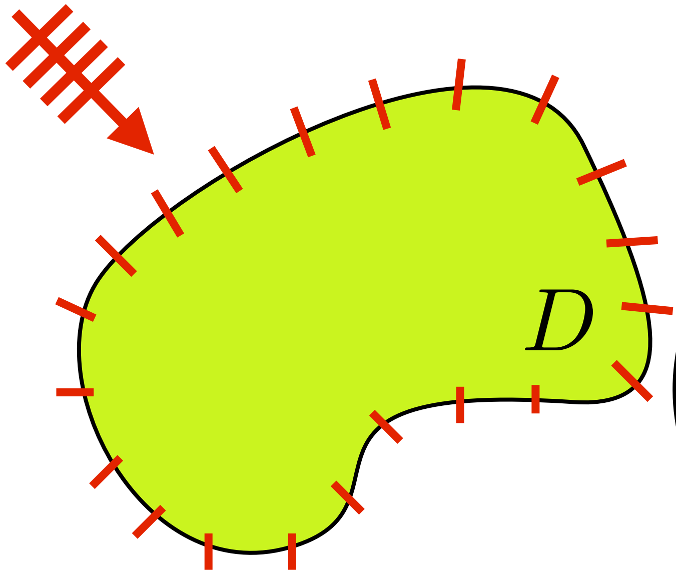

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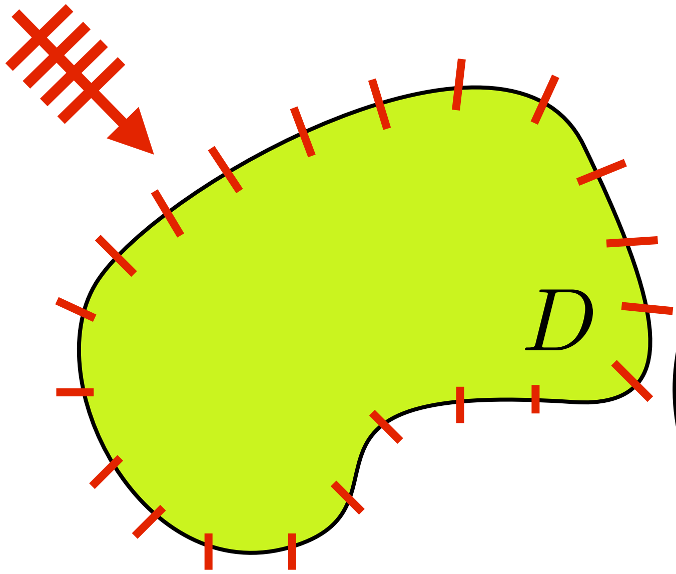

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Problem: the kernel is sharply peaked when x approaches the boundary.

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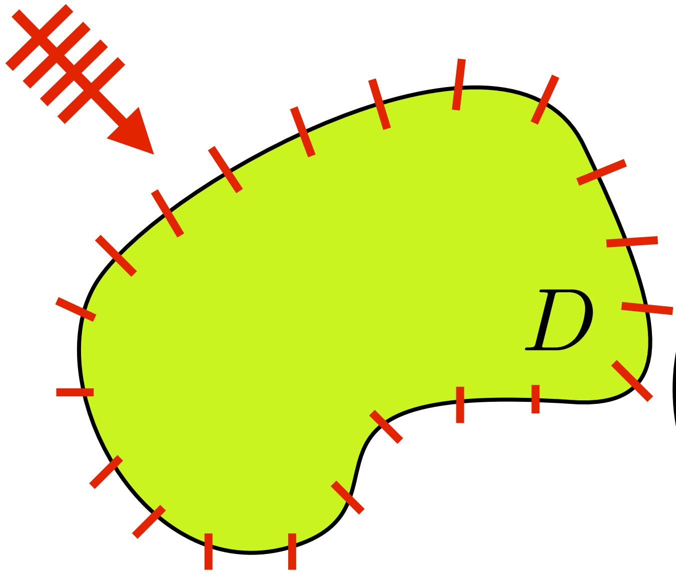

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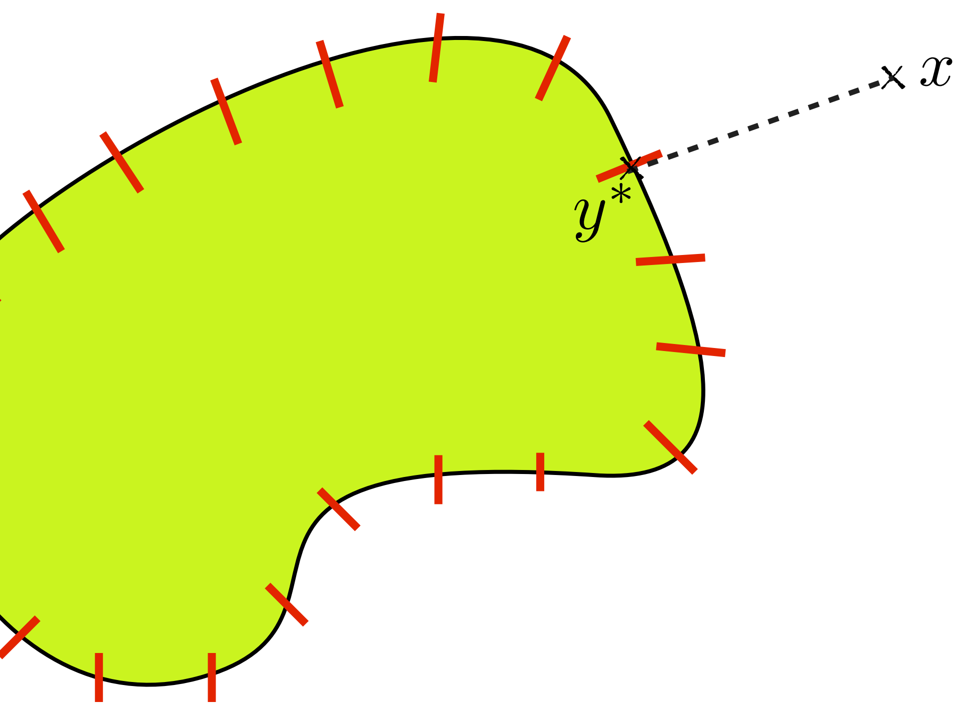

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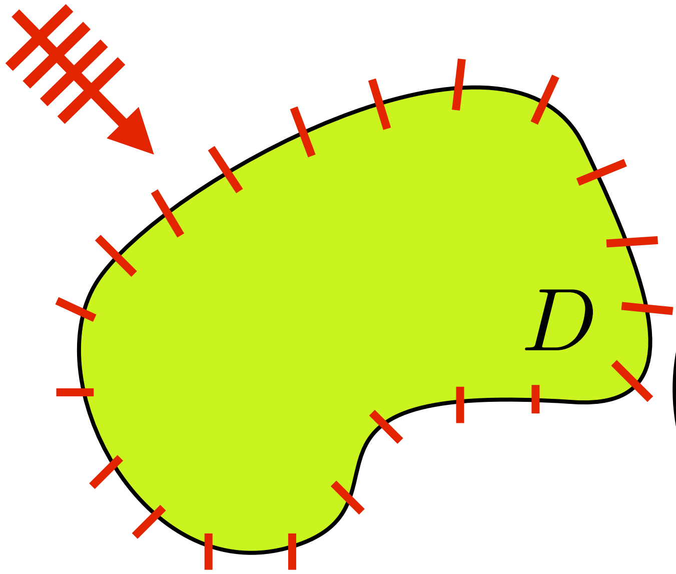

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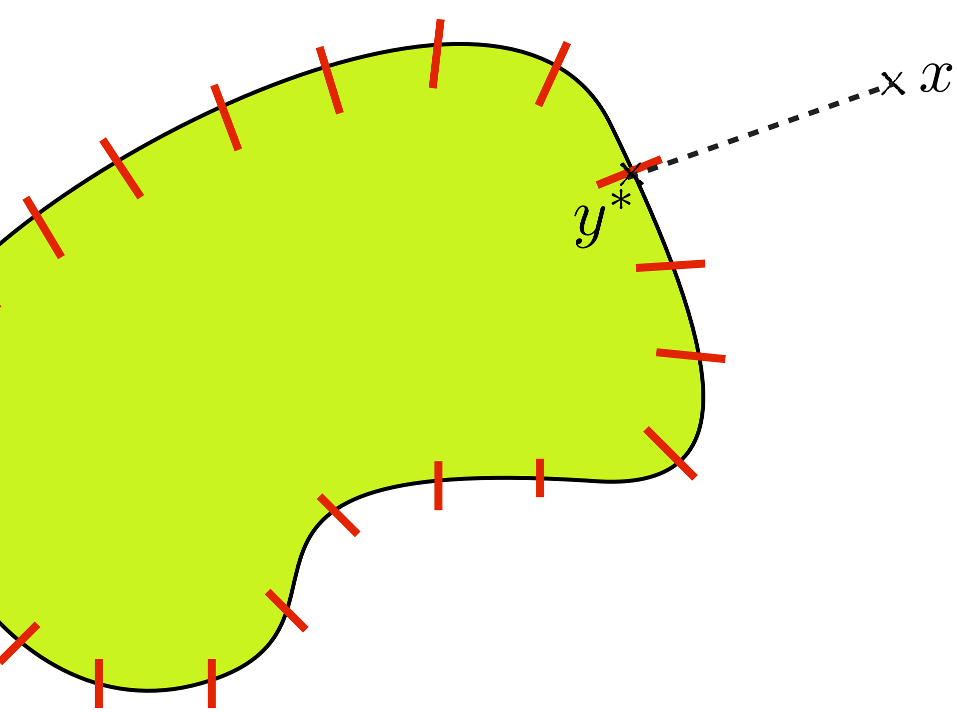

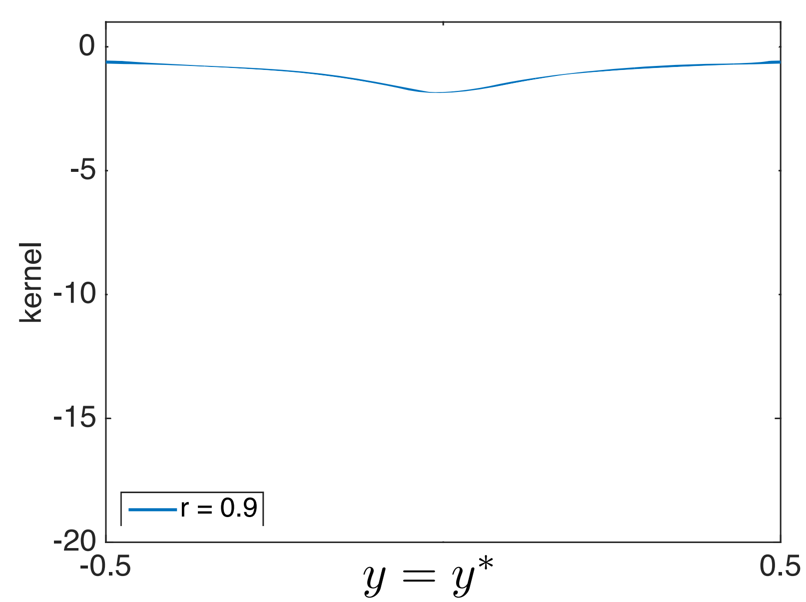

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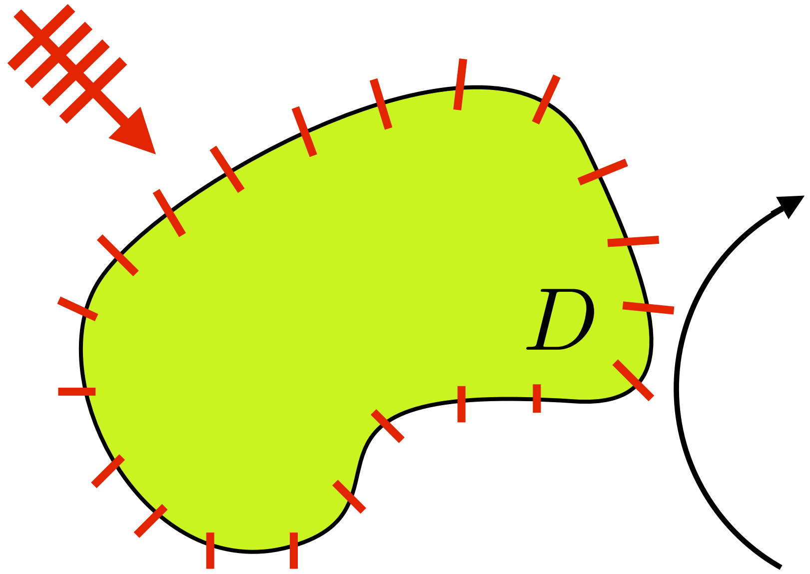

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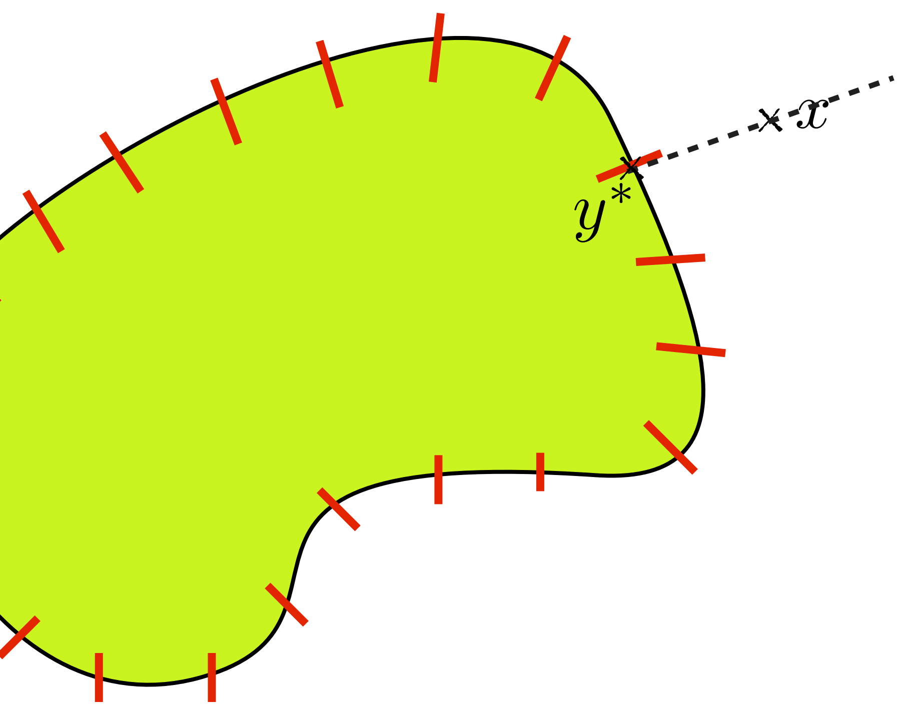

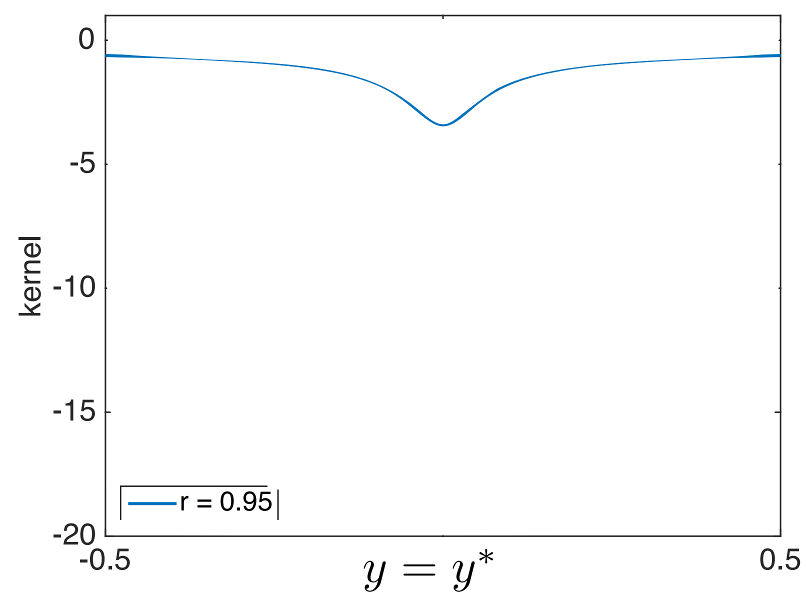

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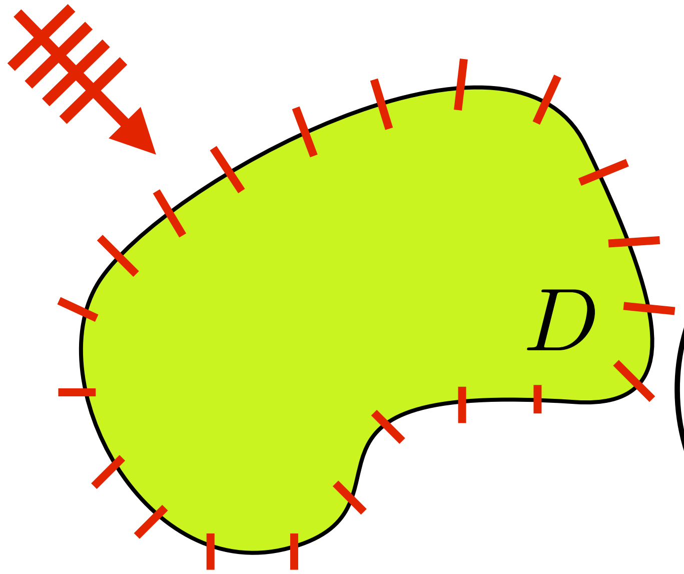

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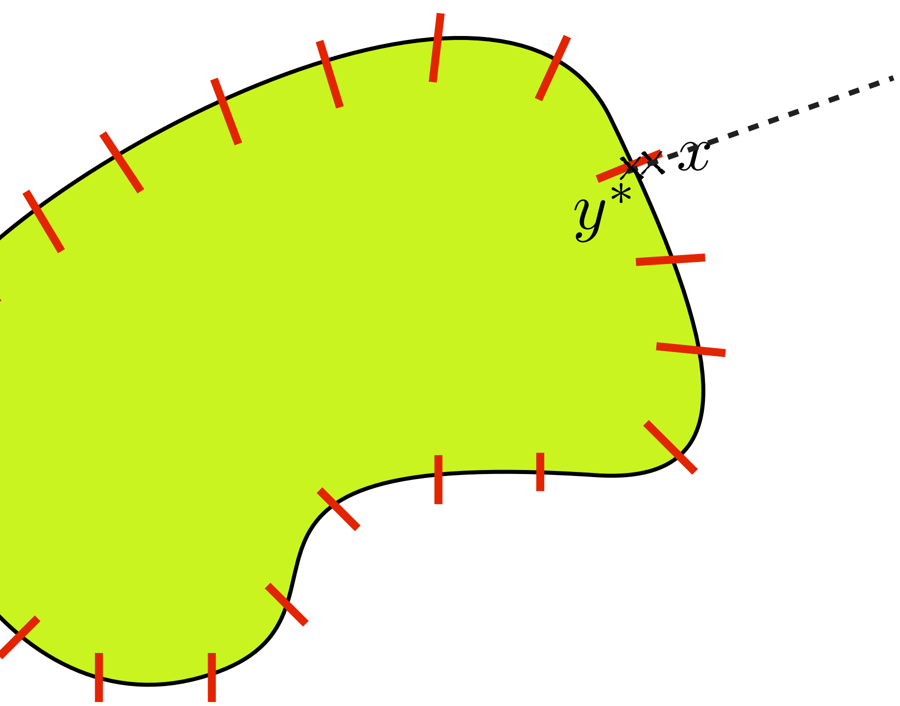

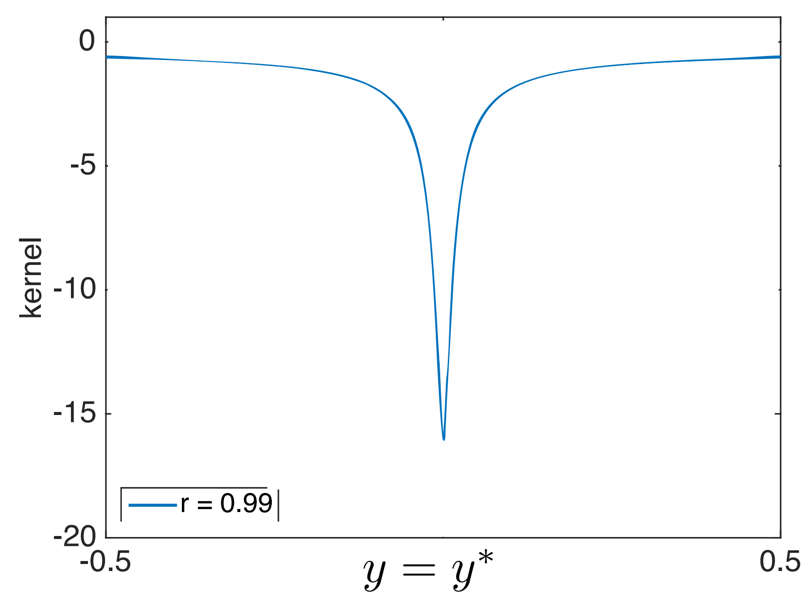

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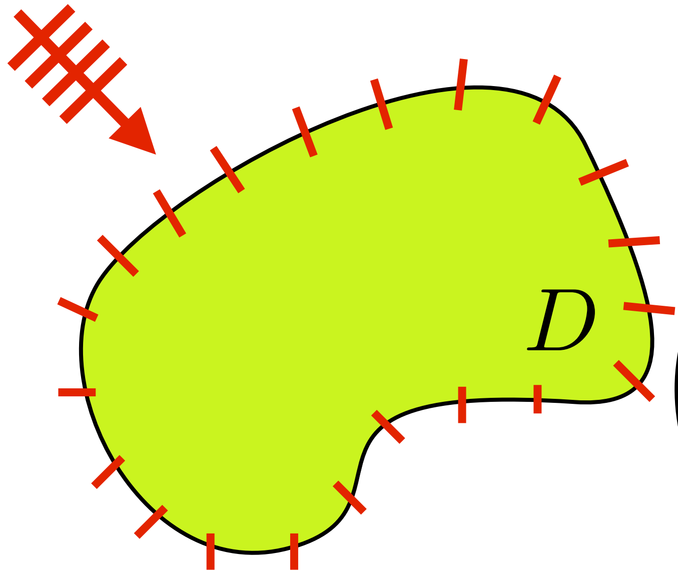

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Numerically

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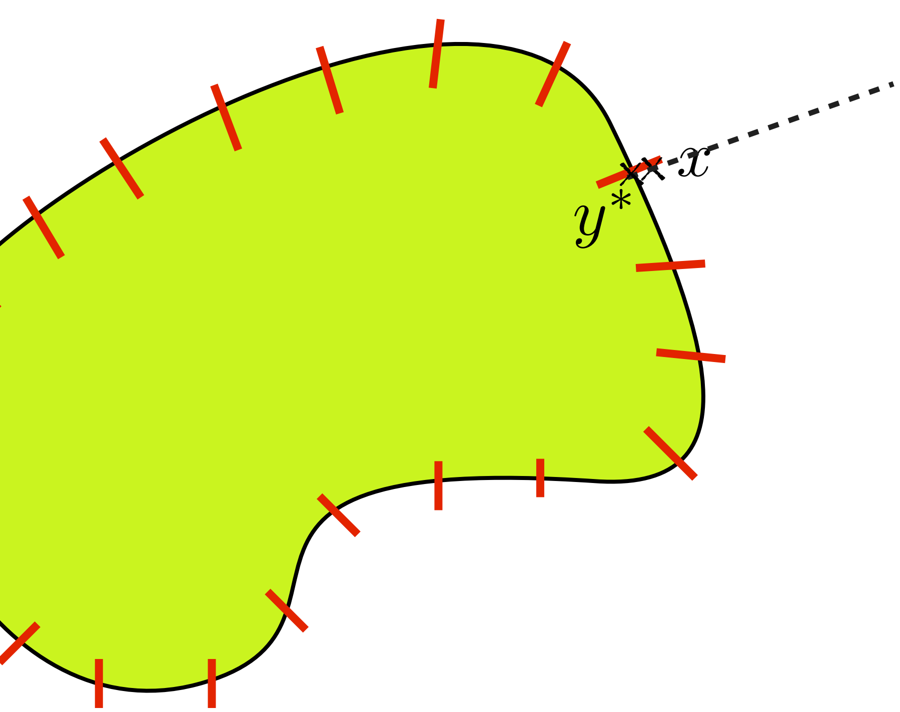
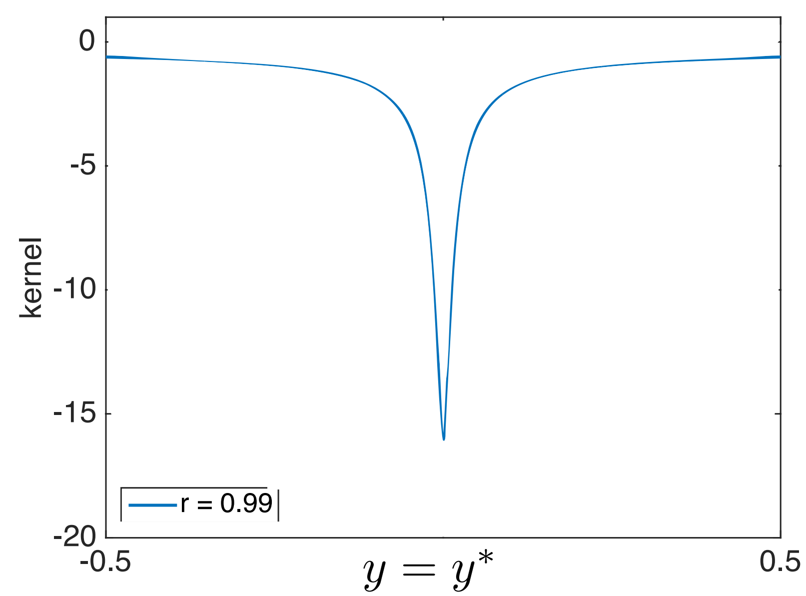
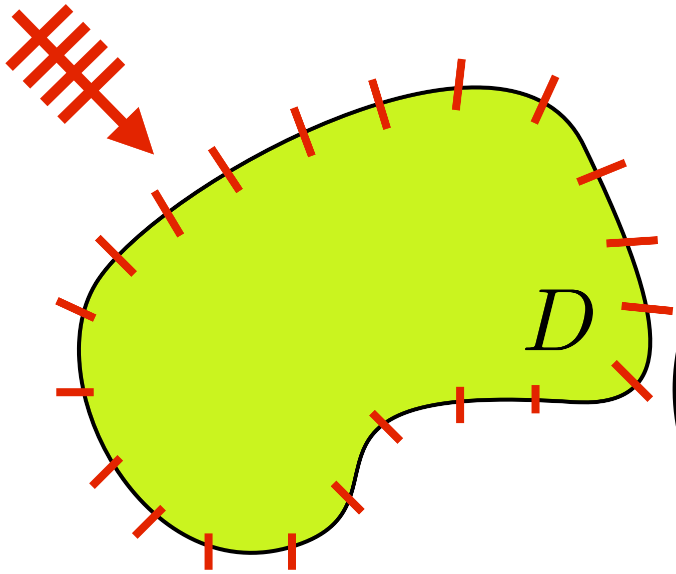
Problem: the kernel is sharply peaked when x approaches the boundary.





Nearly singular integral

Boundary integral methods



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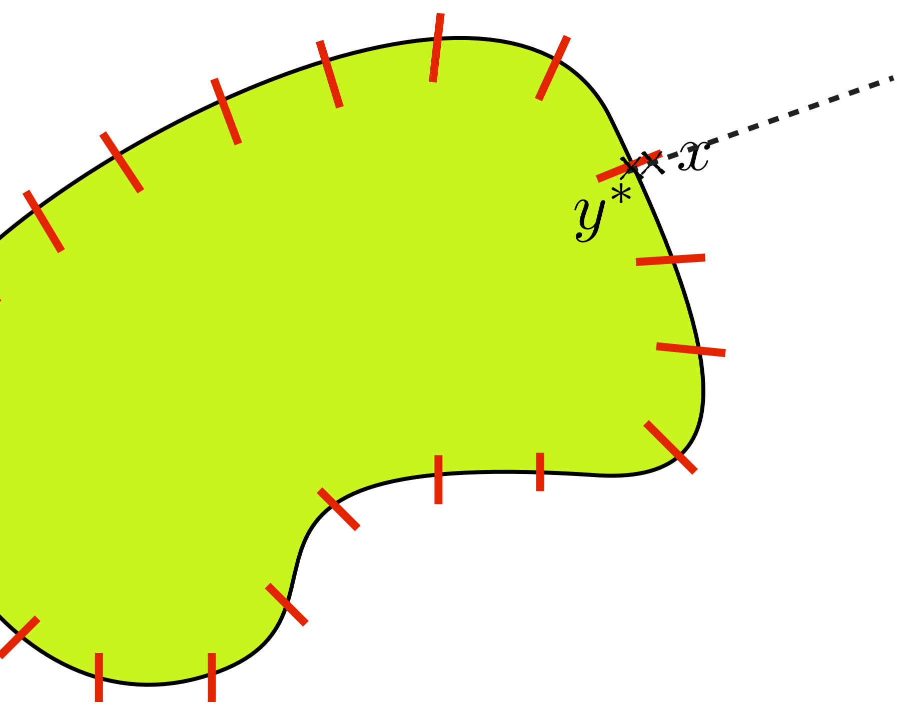

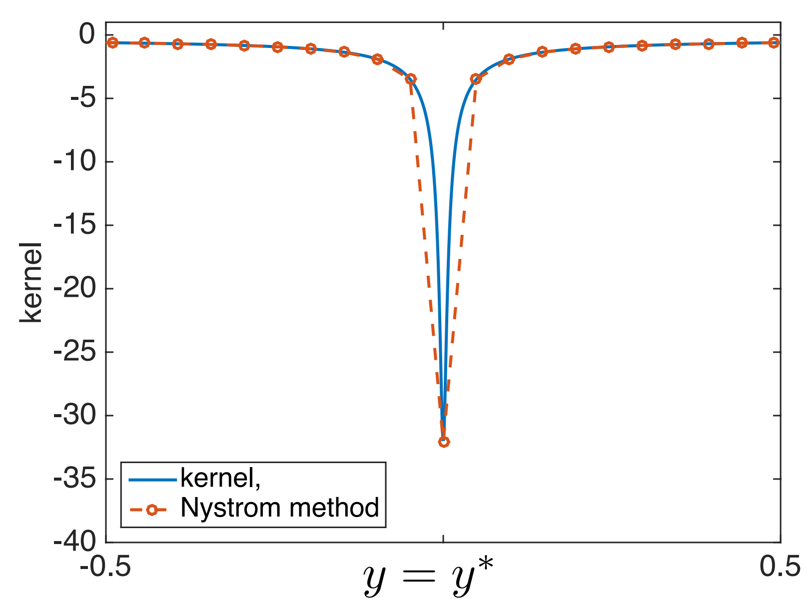

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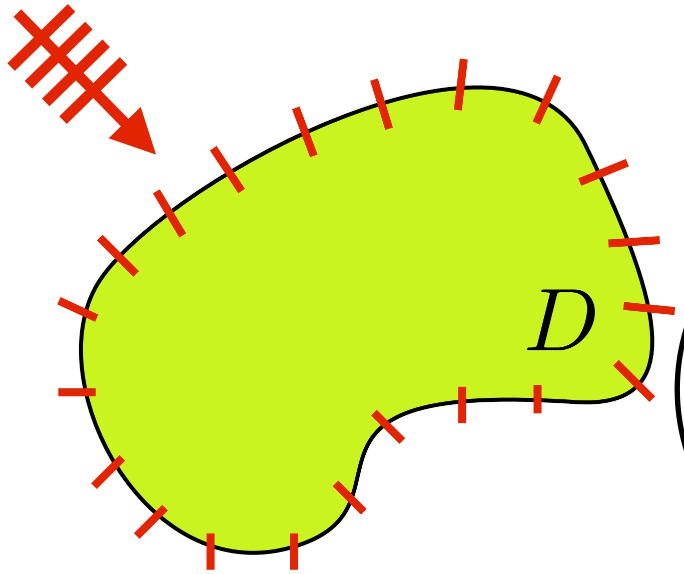

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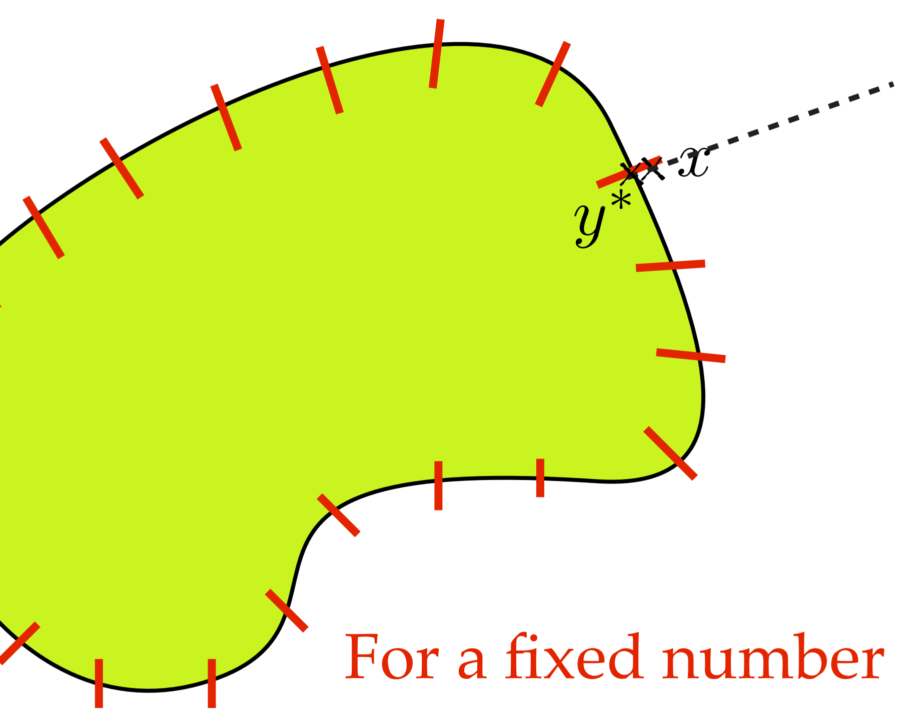
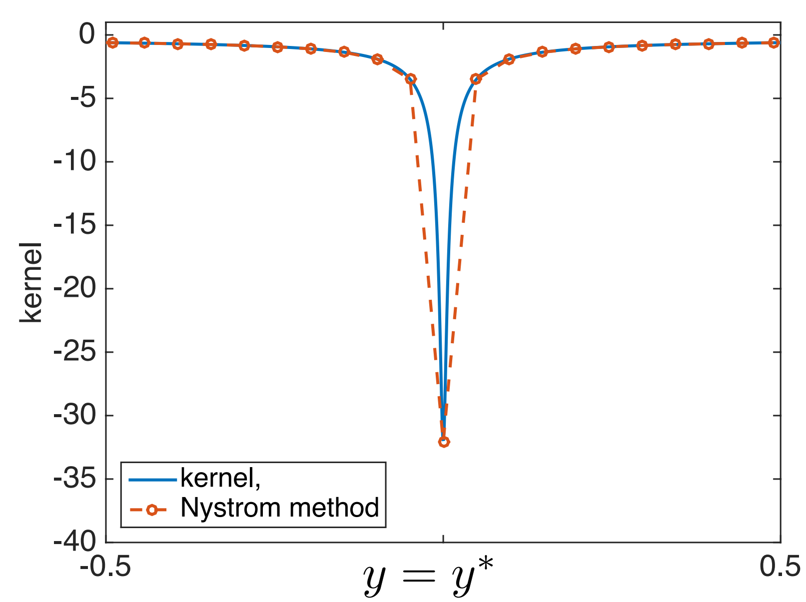


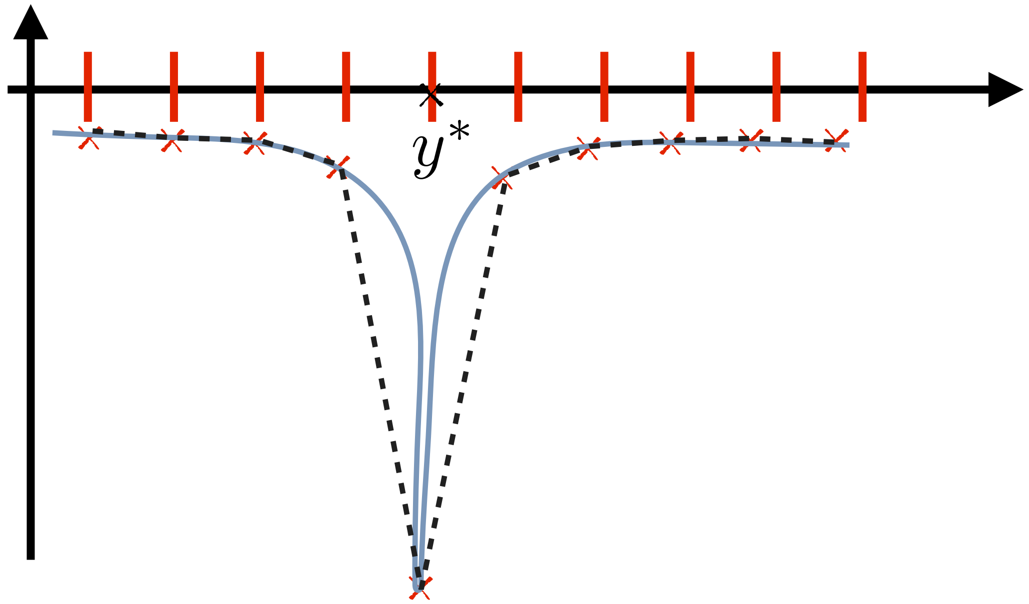


Nearly singular integral

For a fixed number of quadrature points, O(1) error.

Barnett (2014).

How to address this error ?



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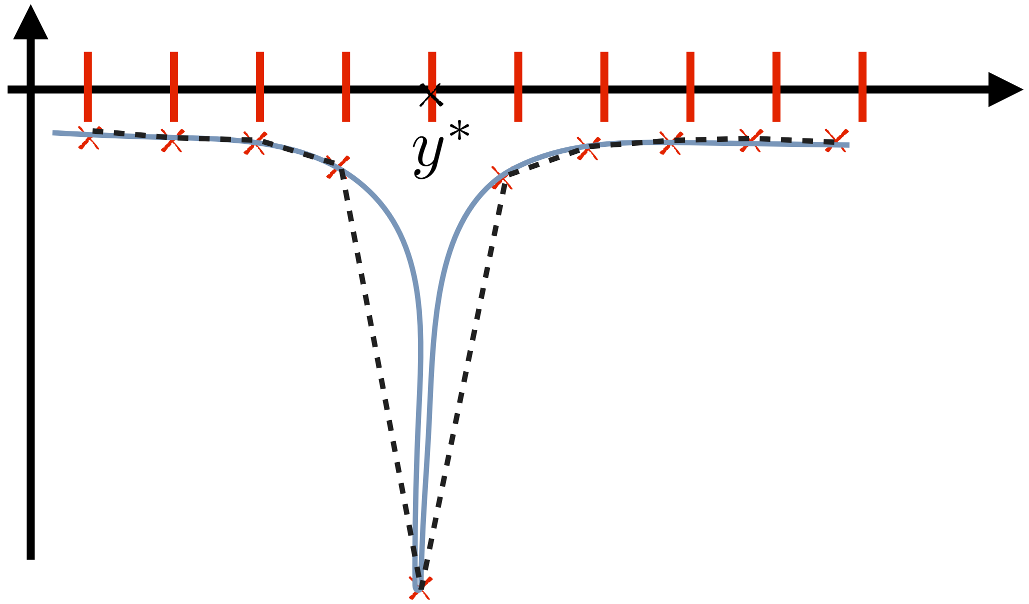

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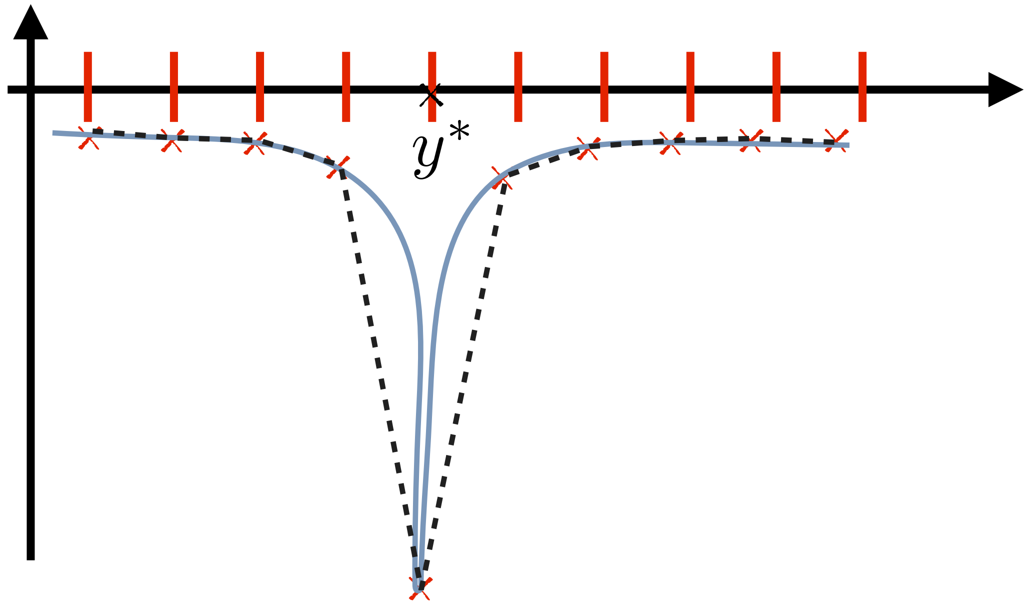

How to address this error ?

1) increase the number of quadrature points
   - ✘ bigger linear system to solve

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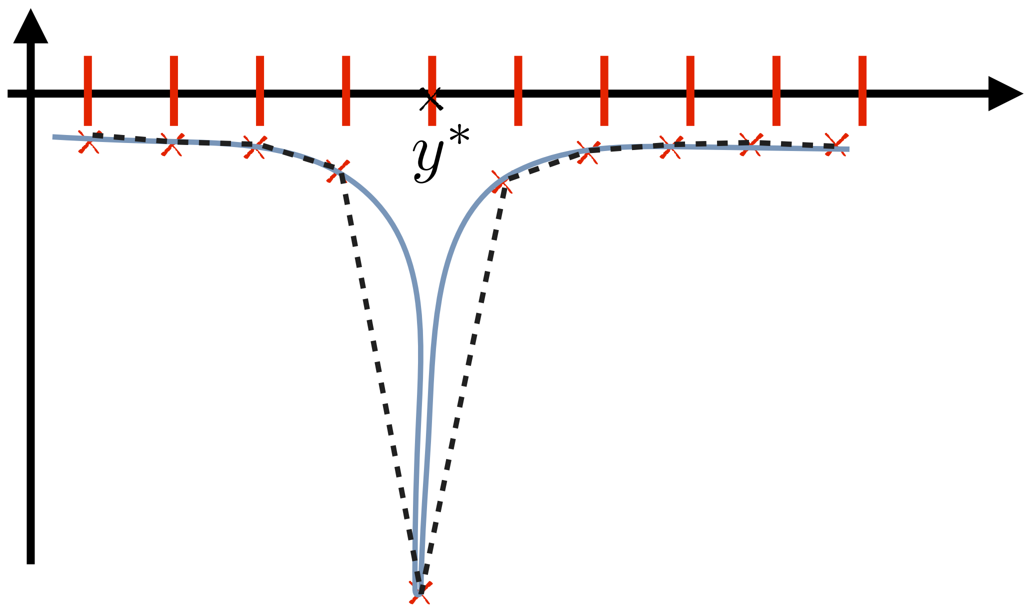

How to address this error ?

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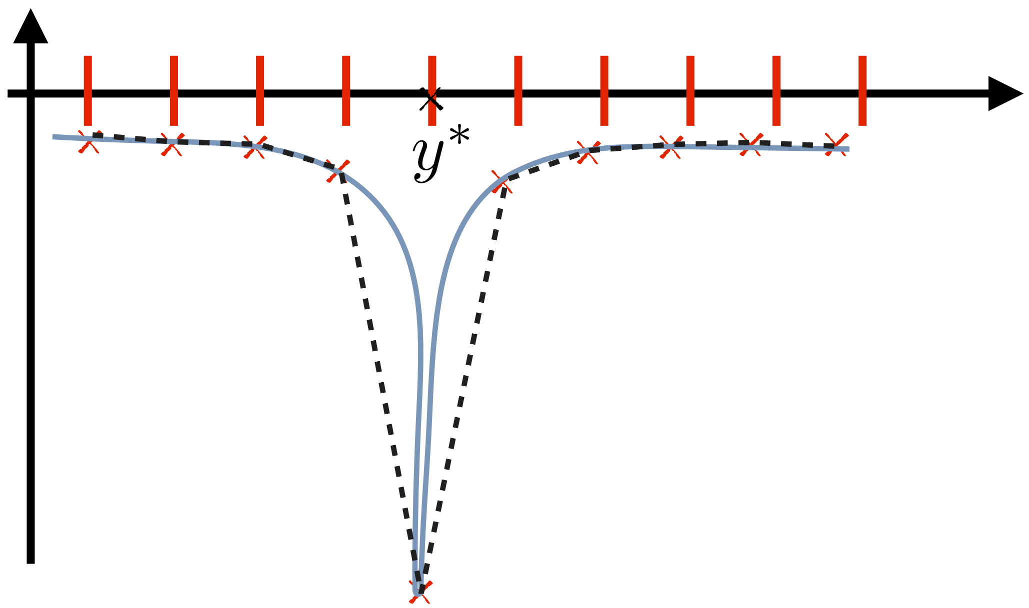

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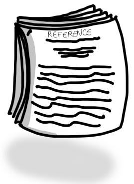

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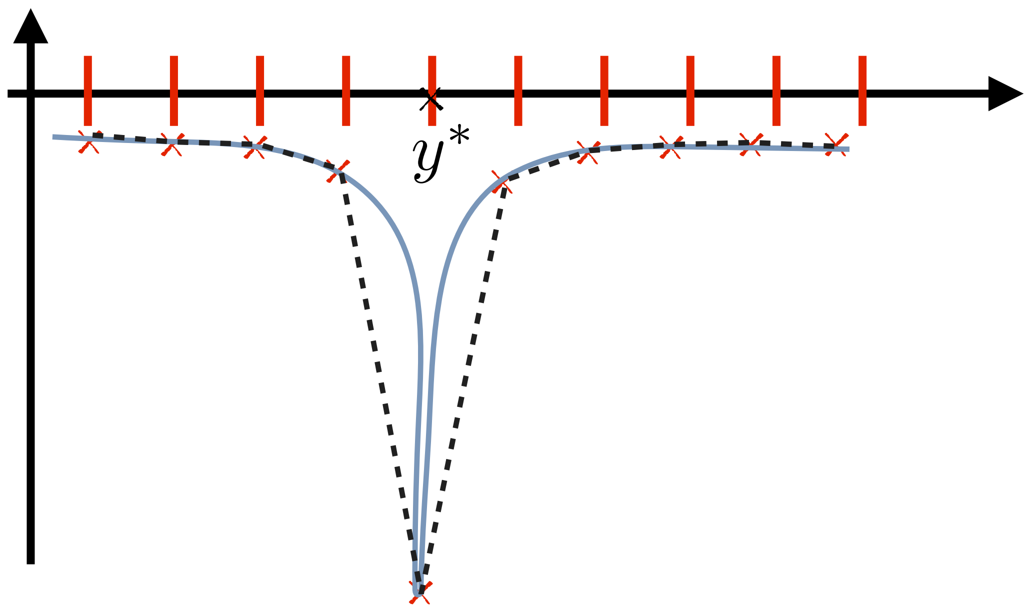

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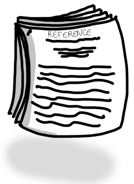

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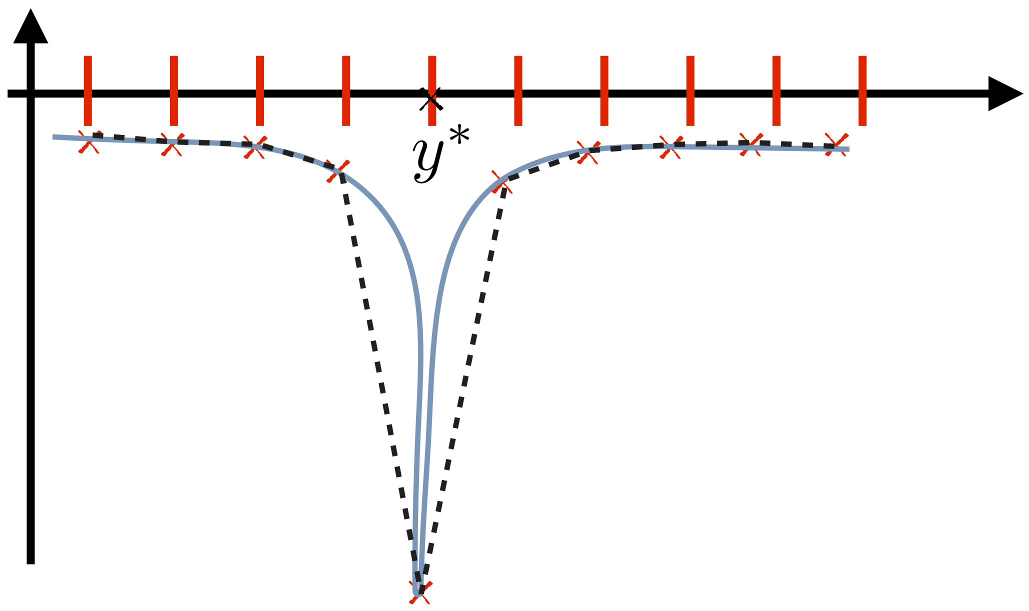

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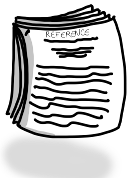

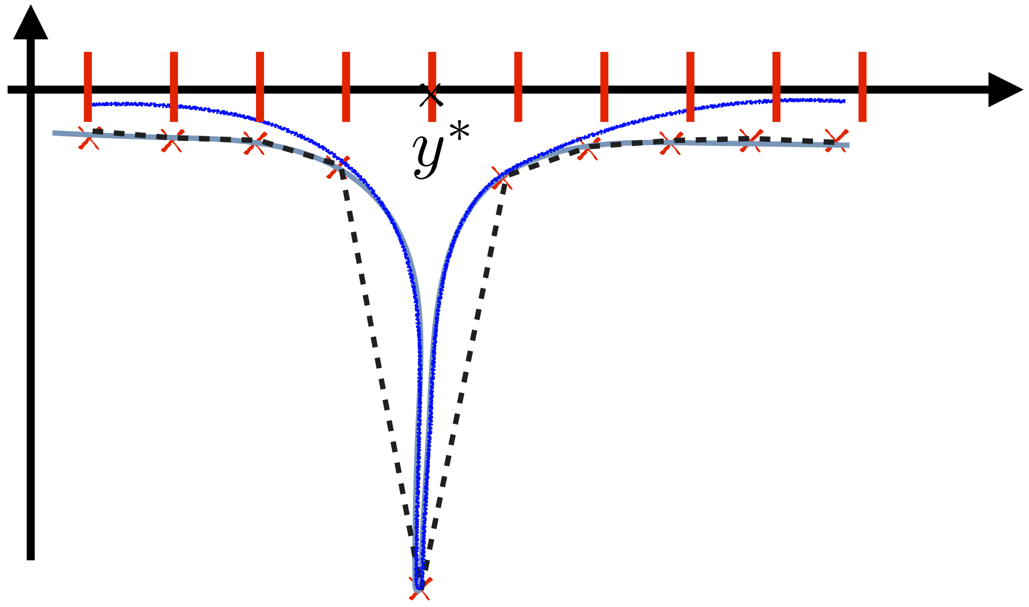

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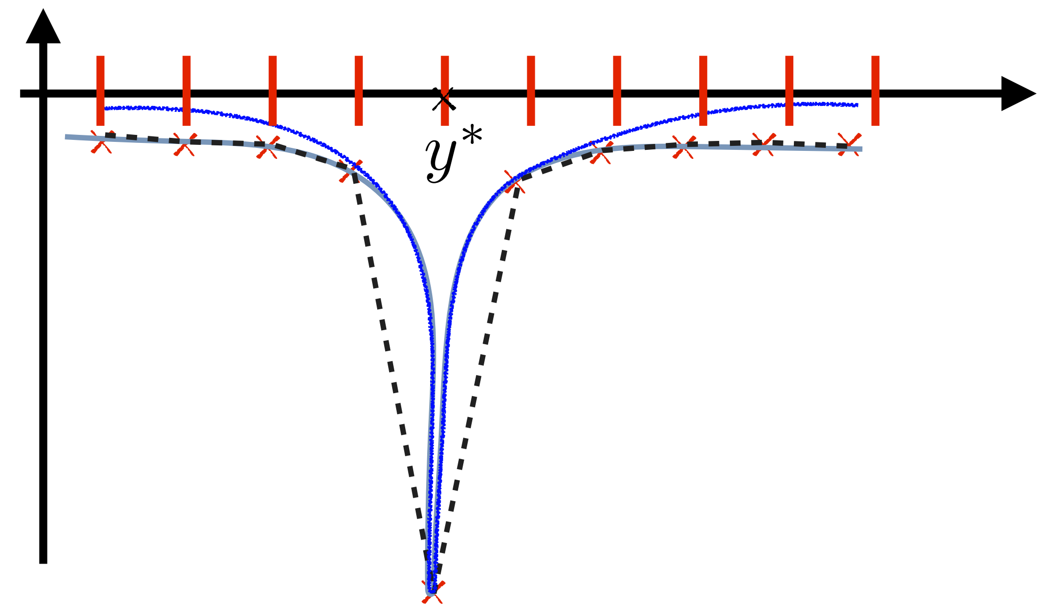

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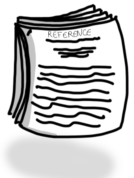

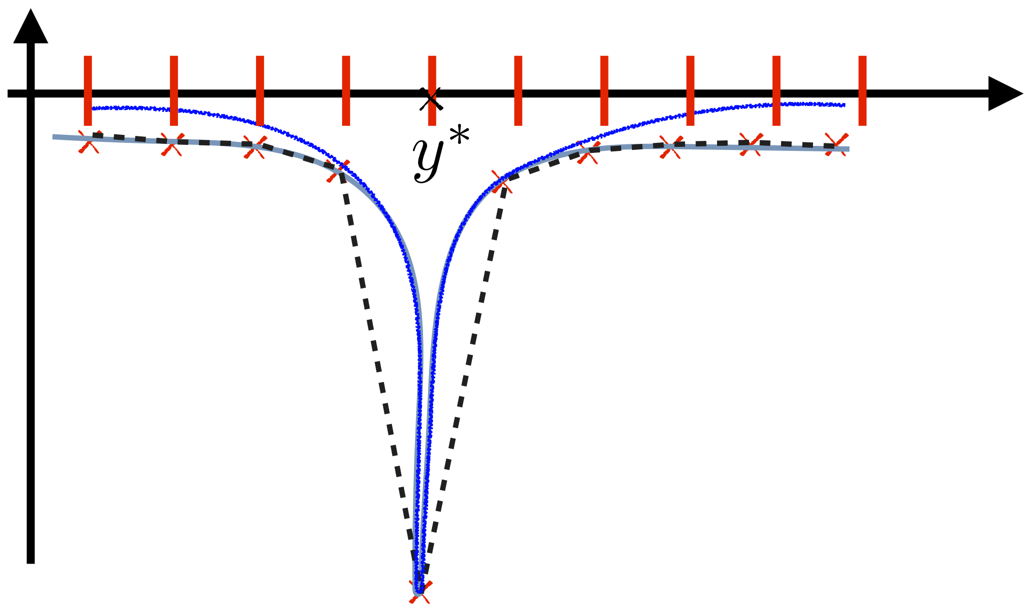

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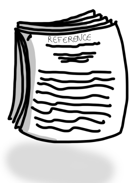

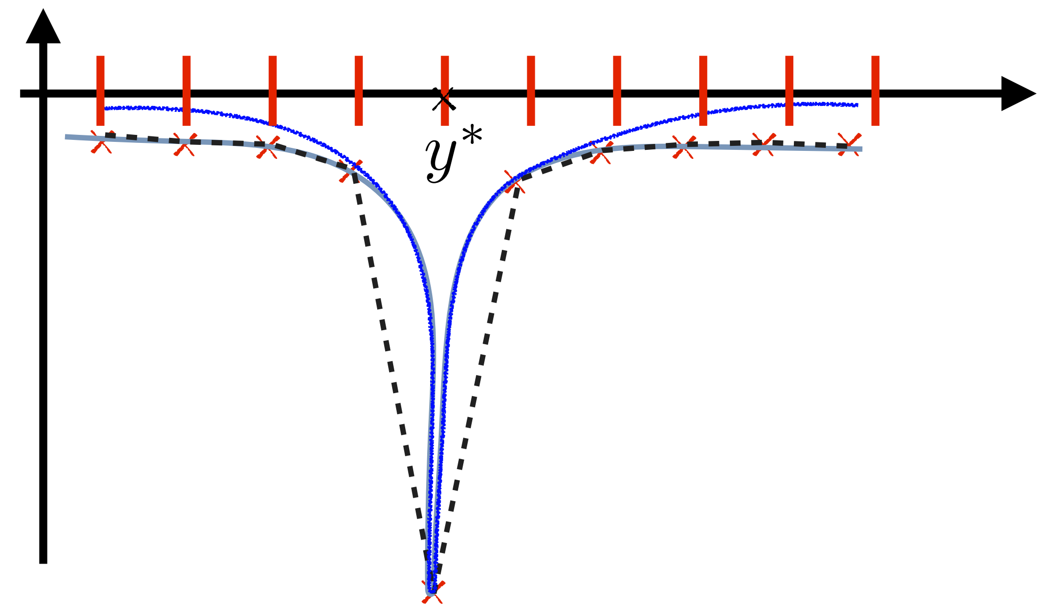

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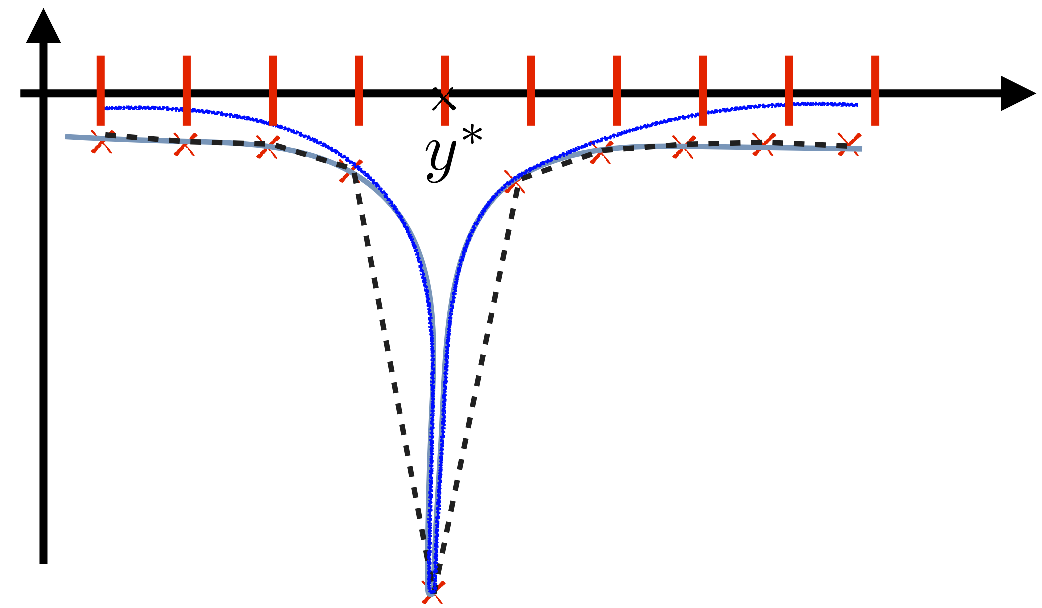

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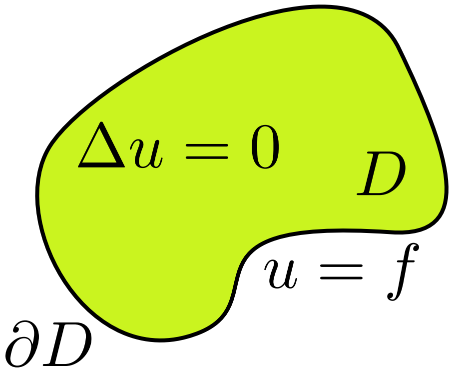
Spectral computation

Outline

- Introduction
- Subtraction techniques for Laplace's equation
- Extension to Helmholtz
- Conclusion

Subtraction technique for Laplace (I)



Subtraction technique for Laplace (I)

The solution of the interior Dirichlet Laplace problem can be represented as

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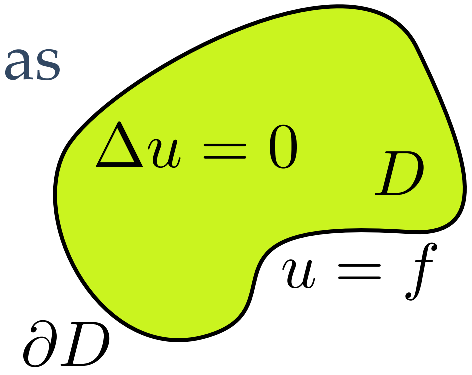

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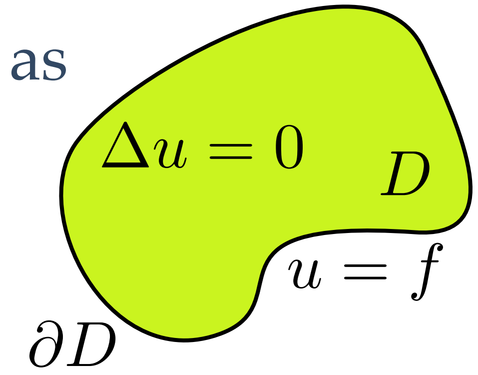

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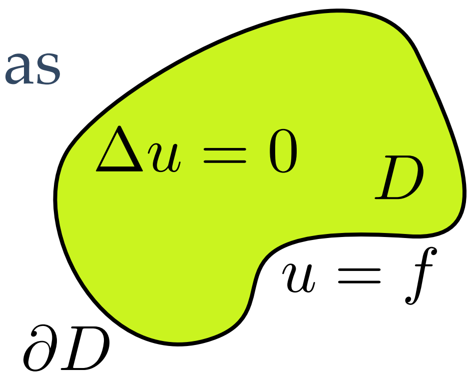

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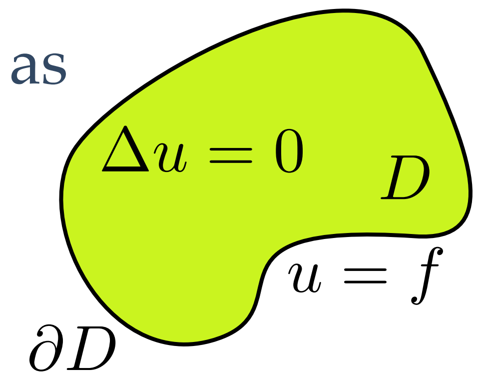

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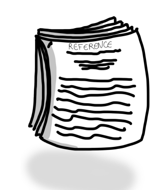

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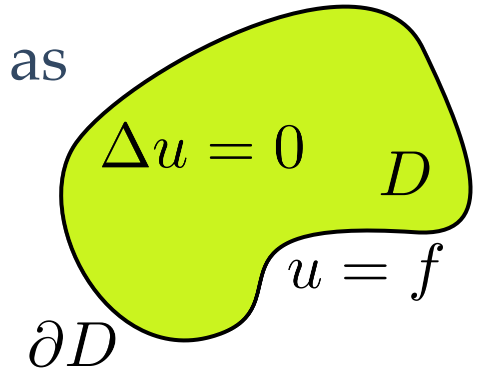

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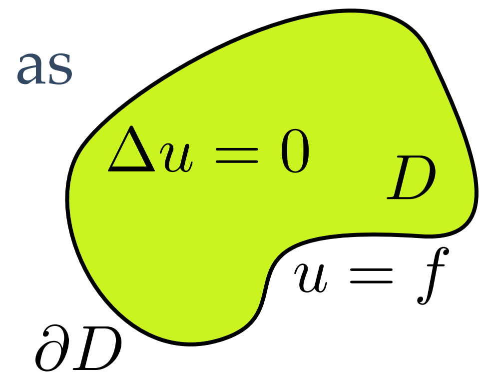

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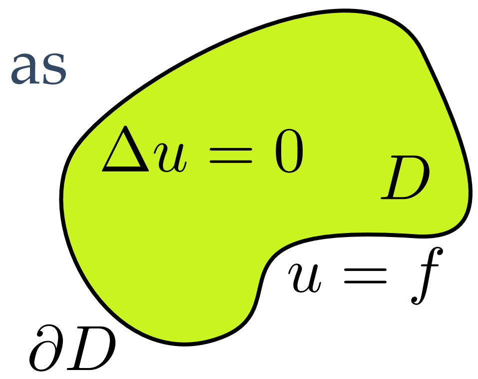

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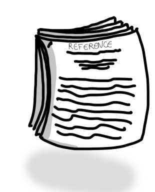

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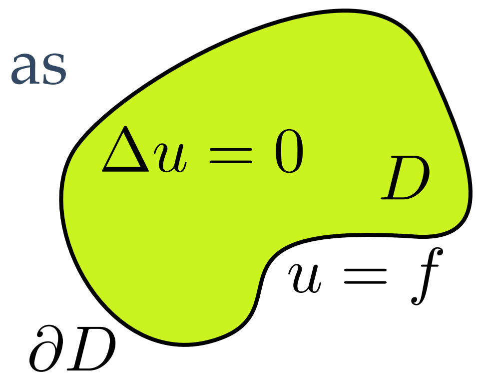

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Subtraction technique for Laplace (II)

Test using Periodic Trapezoid Rule (PTR) with N = 128 for $u(x) = \log|x - x_0|$

Method 1: PTR

$$u(x) = \int_{\partial D} \partial_{n_y} G(x,y)\mu(y)\, d\sigma_y$$

Method 2: PTR + density subtraction

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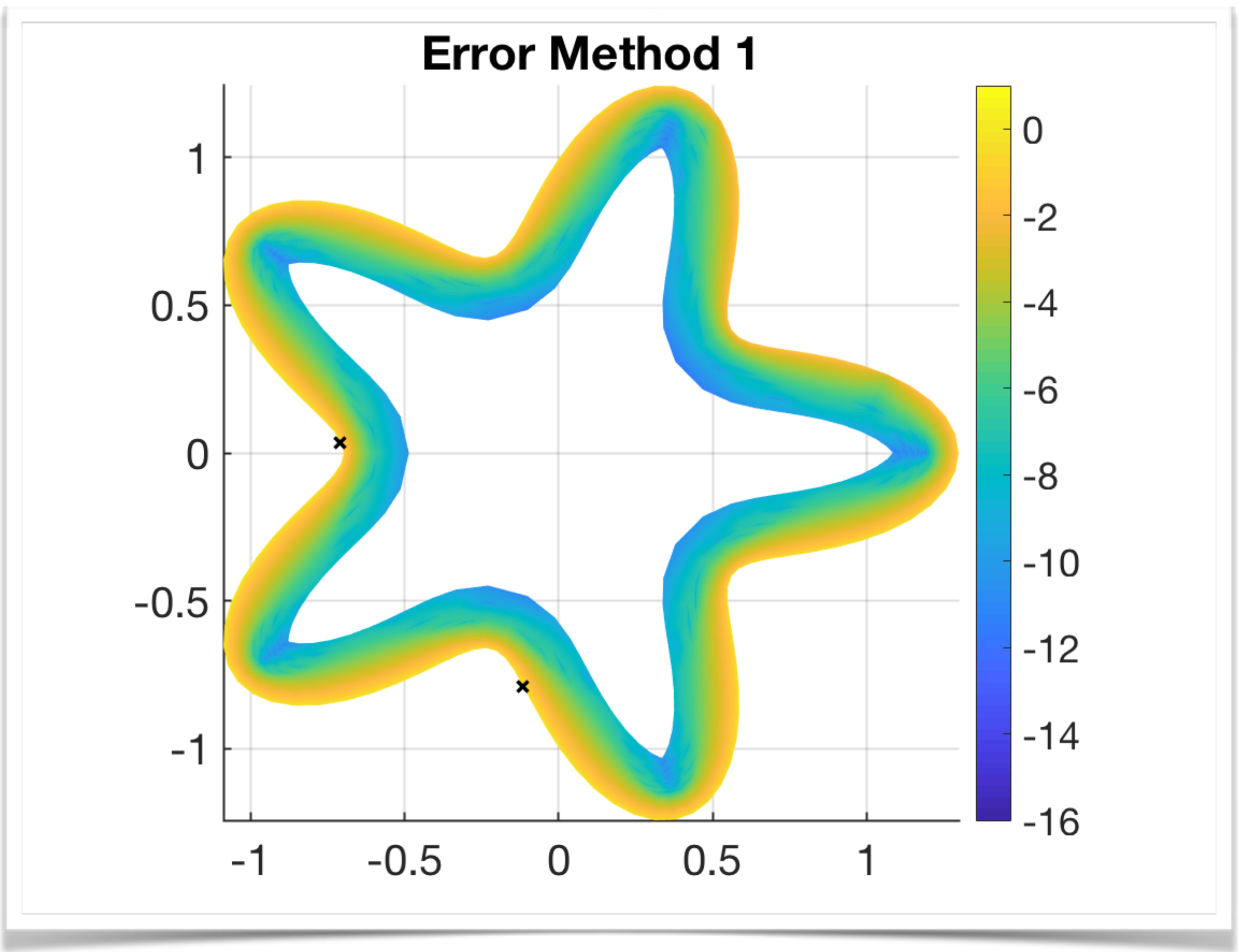

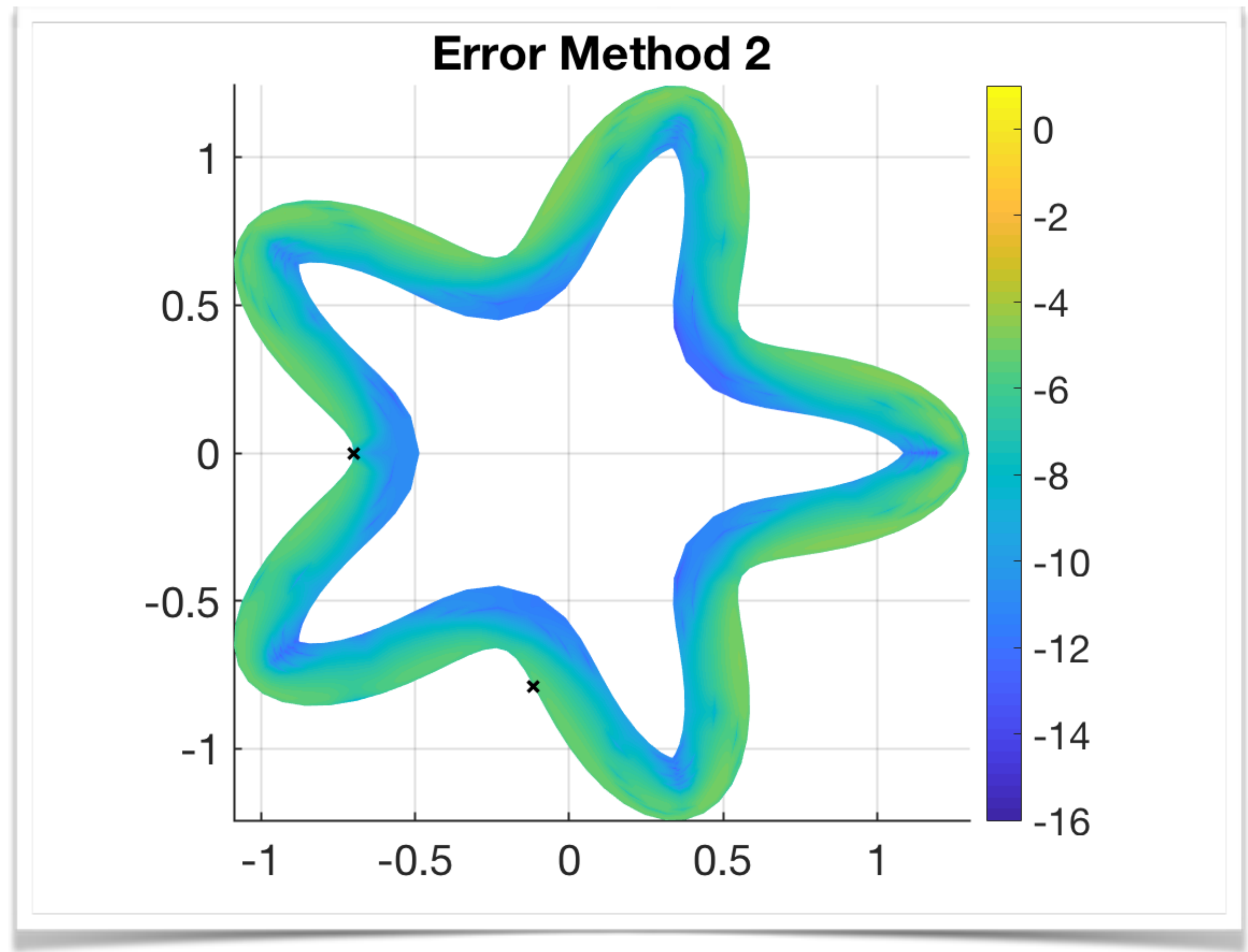

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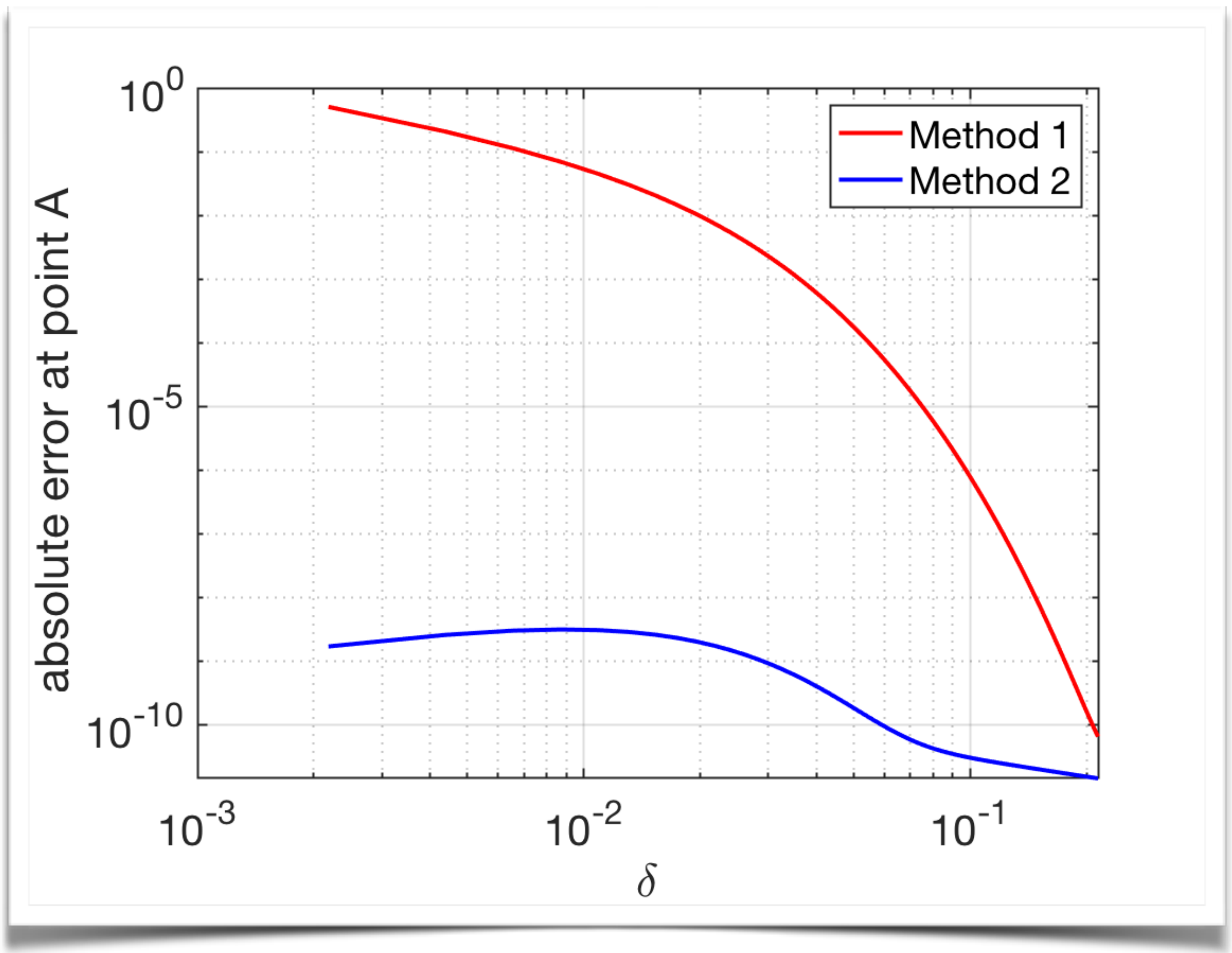
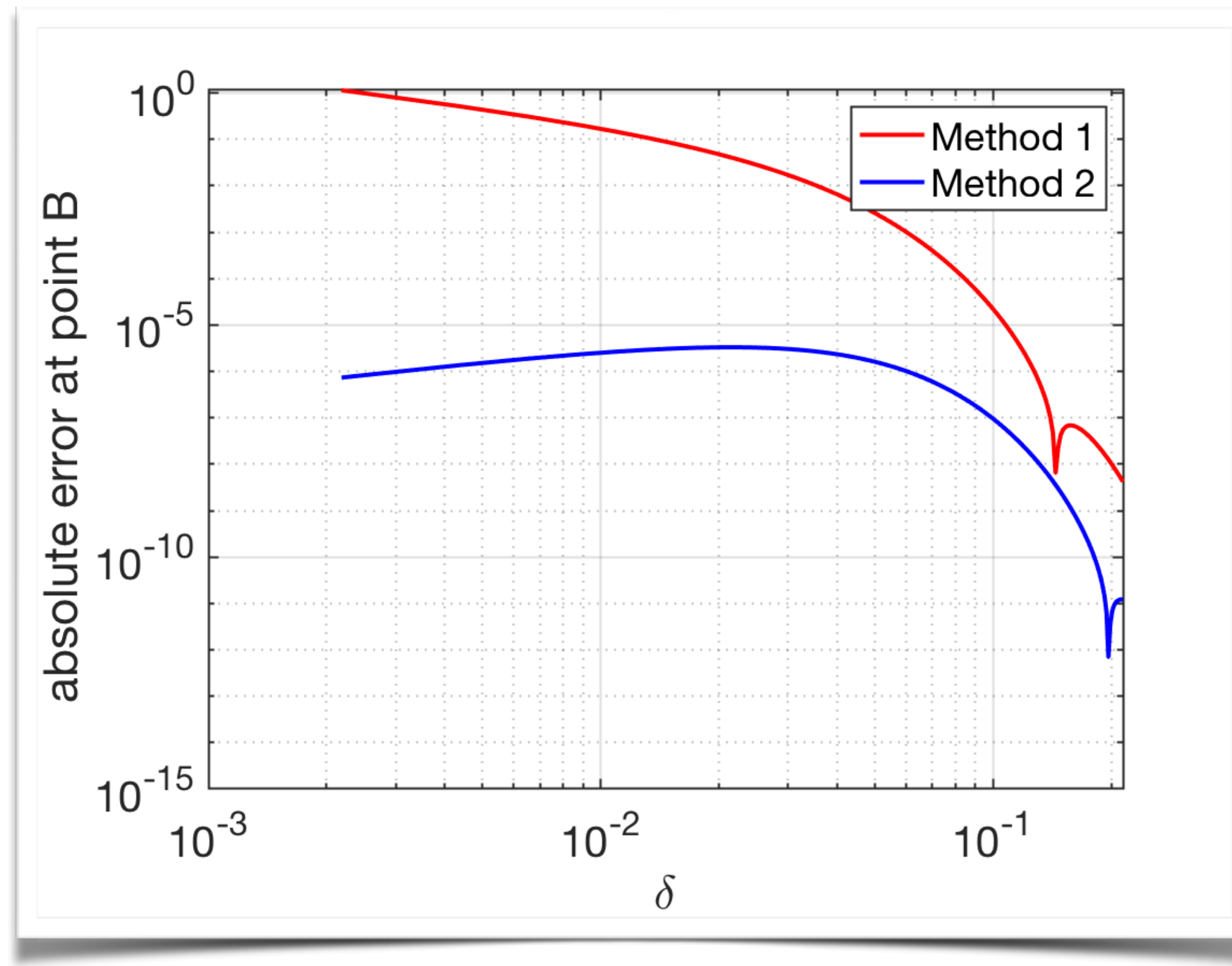
$$u(x) = \int_{\partial D} \partial_{n_y} G(x,y)\mu(y)\, d\sigma_y$$

Method 2: PTR + density subtraction

$$u(x) = \int_{\partial D} \partial_{n_y} G(x,y)[\mu(y) - \mu(x)]\, d\sigma_y - \mu(x)$$





Extensions to Helmholtz

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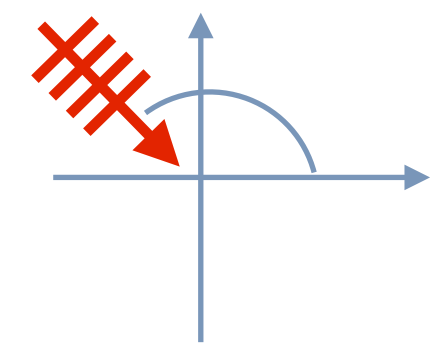
The key is work with solutions of Helmholtz: plane waves $u_d(x) = e^{ik(d \cdot x)}$



Extensions to Helmholtz

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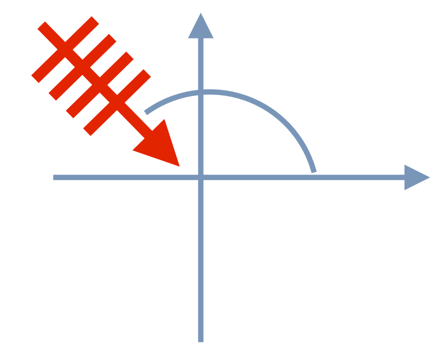
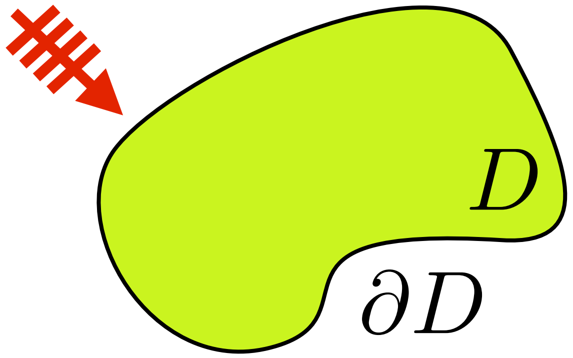


One can show that

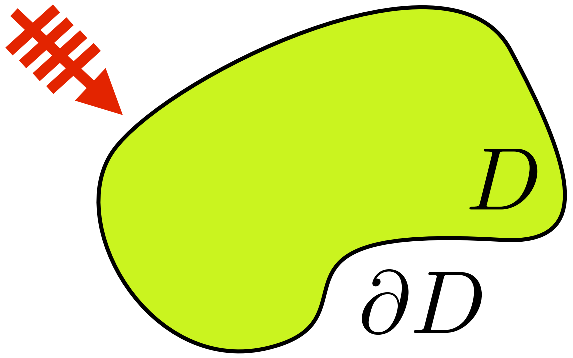
$$\int_{\partial D} [\partial_{n_y} G(x,y) - ik(n_y \cdot d) G(x,y)] e^{ik(d \cdot y)}\, d\sigma_y = \begin{cases} 0 & x \in \mathbb{R}^2 \setminus \bar{D} \\ -\frac{1}{2} e^{ik(d \cdot x)} & x \in \partial D \\ -e^{ik(d \cdot x)} & x \in D \end{cases}$$

Extensions to Helmholtz

$$u(x) = \int_{\partial D} [\partial_{n_y} G(x,y) - ikG(x,y)]\mu(y)\, d\sigma_y$$



Extensions to Helmholtz



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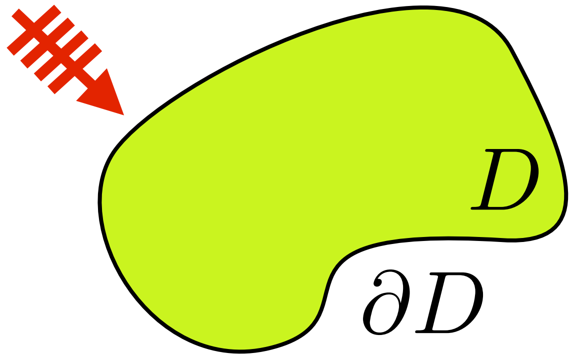

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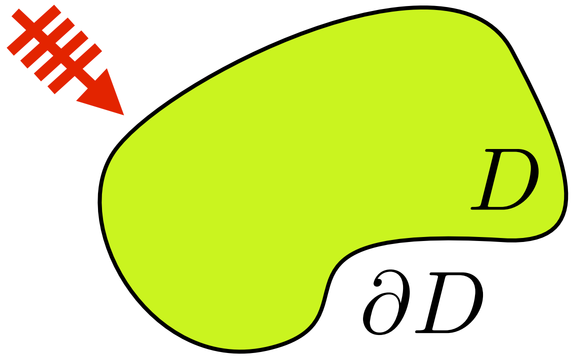

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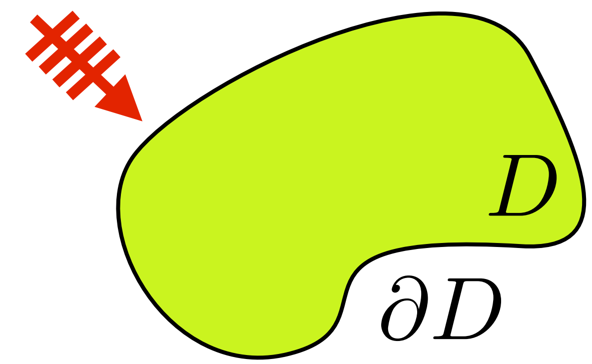
Vanishes at $x = y$

Extensions to Helmholtz



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Plane Wave subtraction

$$\int_{\partial D} [\partial_{n_y} G(x,y) - ik(n_y \cdot n_x)G(x,y)] \left[\mu(y) - \mu(x)e^{ikn_x \cdot (y-x)}\right] d\sigma_y$$

$$+\mu(x)e^{ikn_x \cdot x} \int_{\partial D} [\partial_{n_y} G(x,y) - ik(n_y \cdot n_x)G(x,y)]e^{ikn_x \cdot (y)}\, d\sigma_y$$

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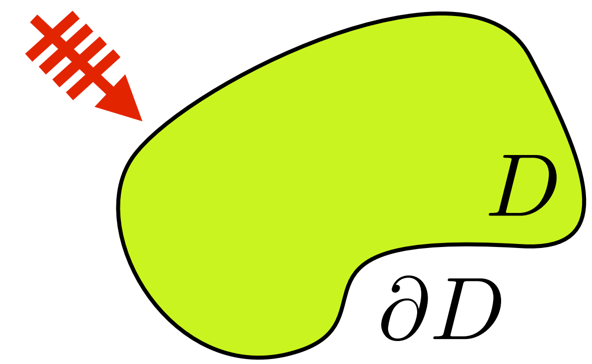

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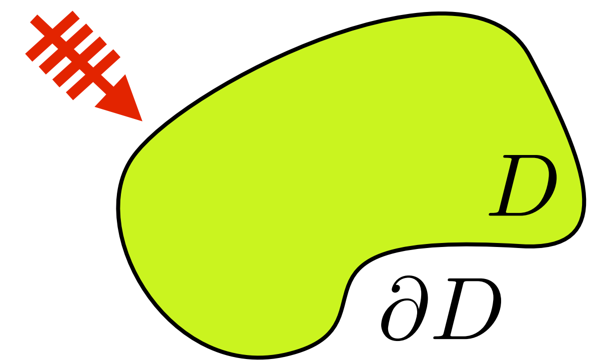

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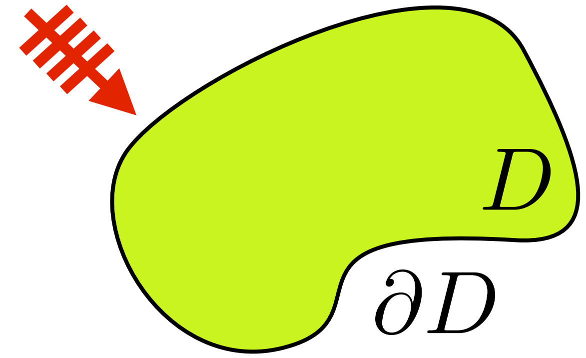
Using new identity

Extensions to Helmholtz



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Using new identity

$$\boxed{\begin{aligned} u(x) = &\int_{\partial D} [\partial_{n_y} G(x,y) - ik(n_y \cdot n_x)G(x,y)] \left[\mu(y) - \mu(x)e^{ikn_x \cdot (y-x)}\right] d\sigma_y \\ &- i\int_{\partial D} [k - k(n_y \cdot n_x)]G(x,y)\mu(y)\, d\sigma_y \end{aligned}}$$

Subtraction technique for Helmholtz

Test using Periodic Trapezoid Rule (PTR) with N = 256 for $u(x) := \frac{i}{4} H_0^{(1)}(k|x - x_0|)$ $k = 15$

Method 1: PTR

Method 2: PTR + PW subtraction

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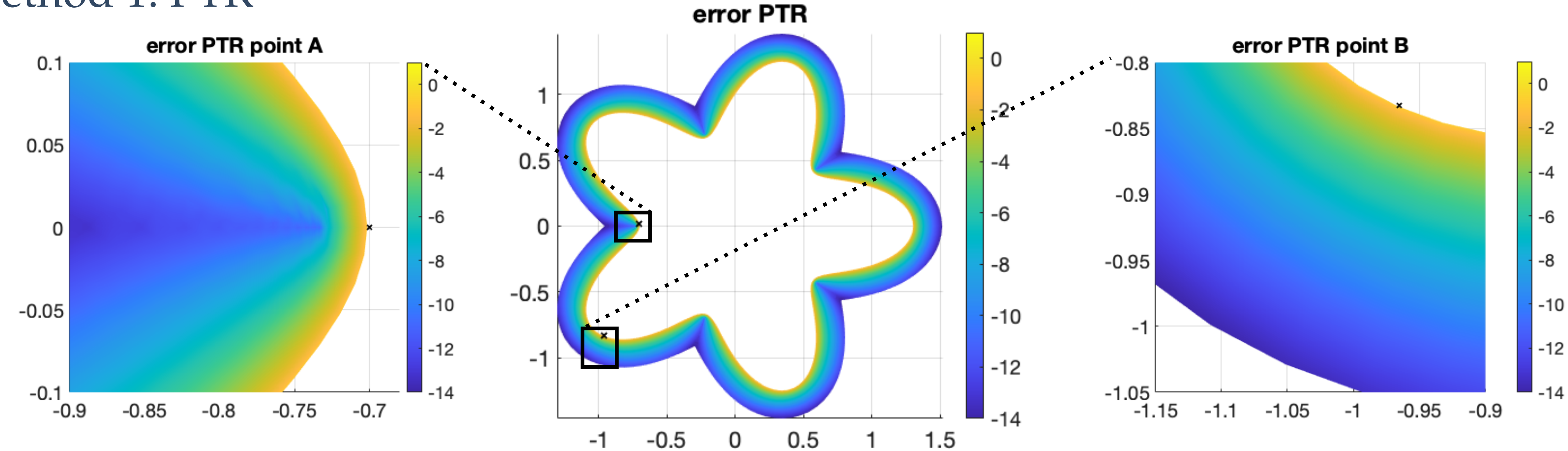
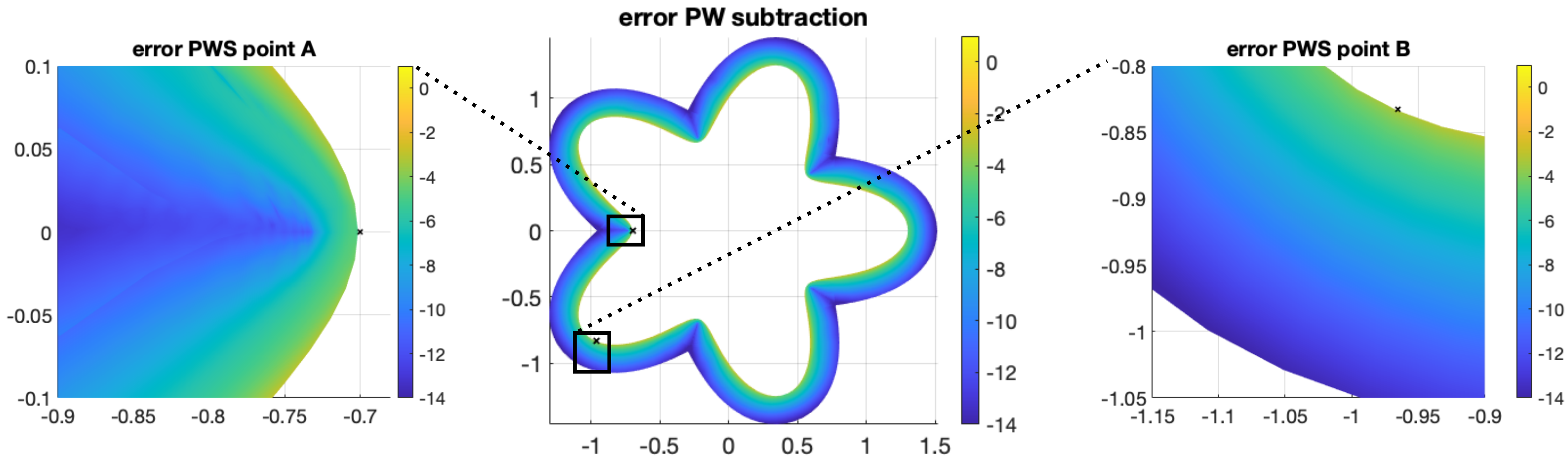


Method 2: PTR + PW subtraction



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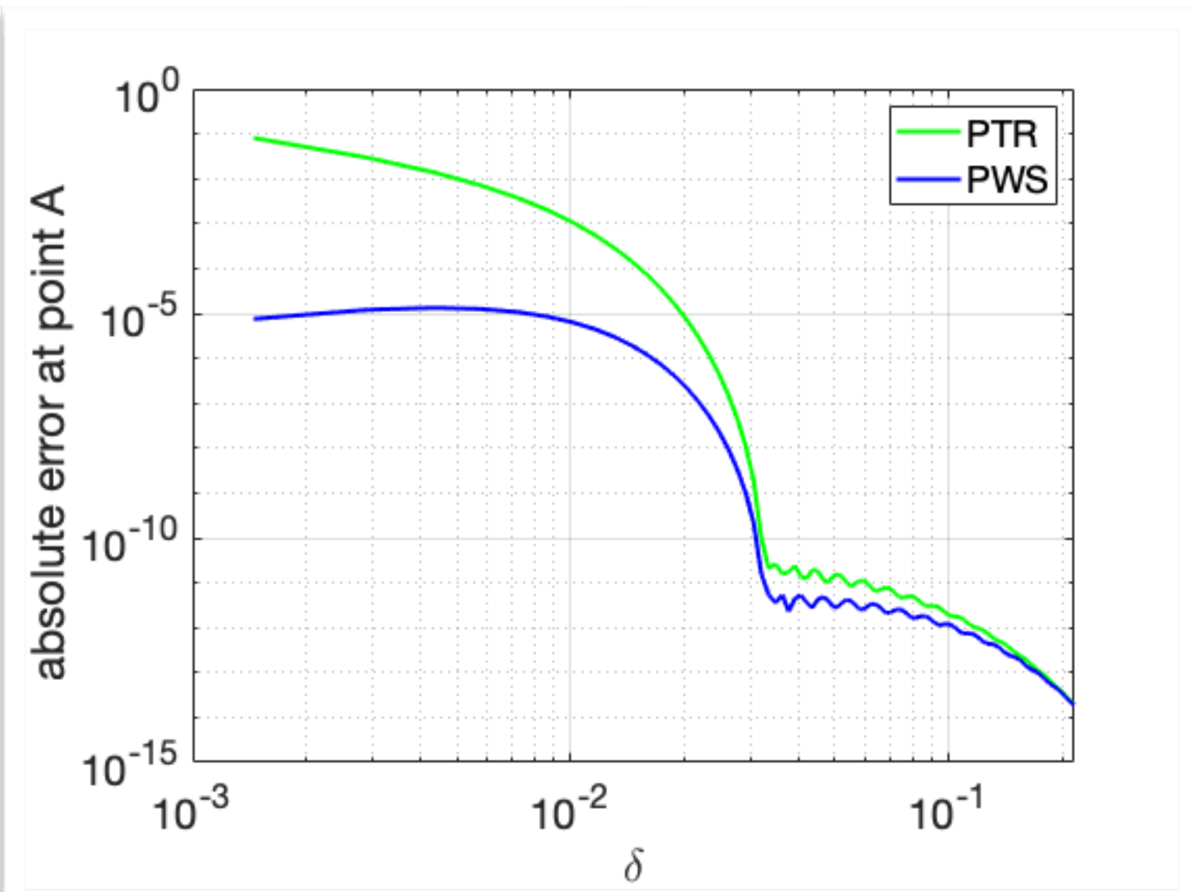
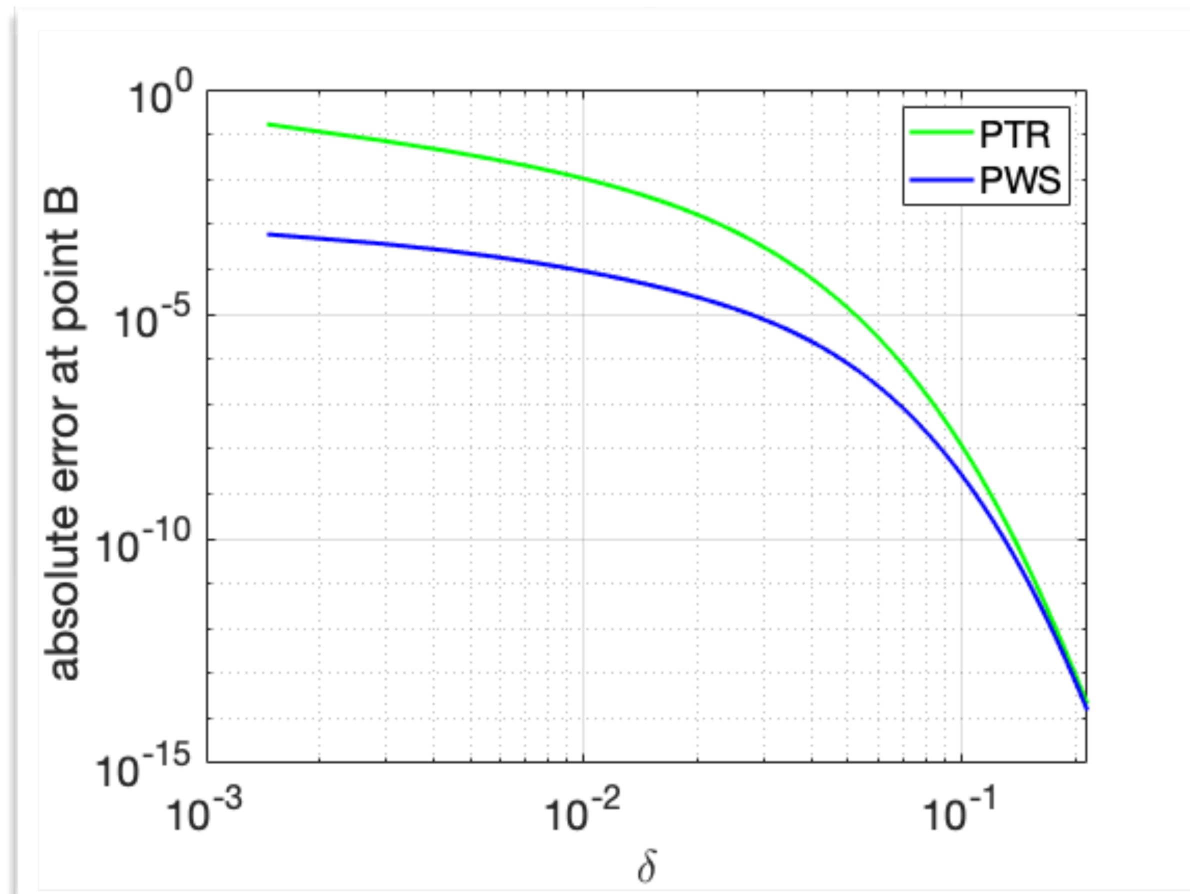
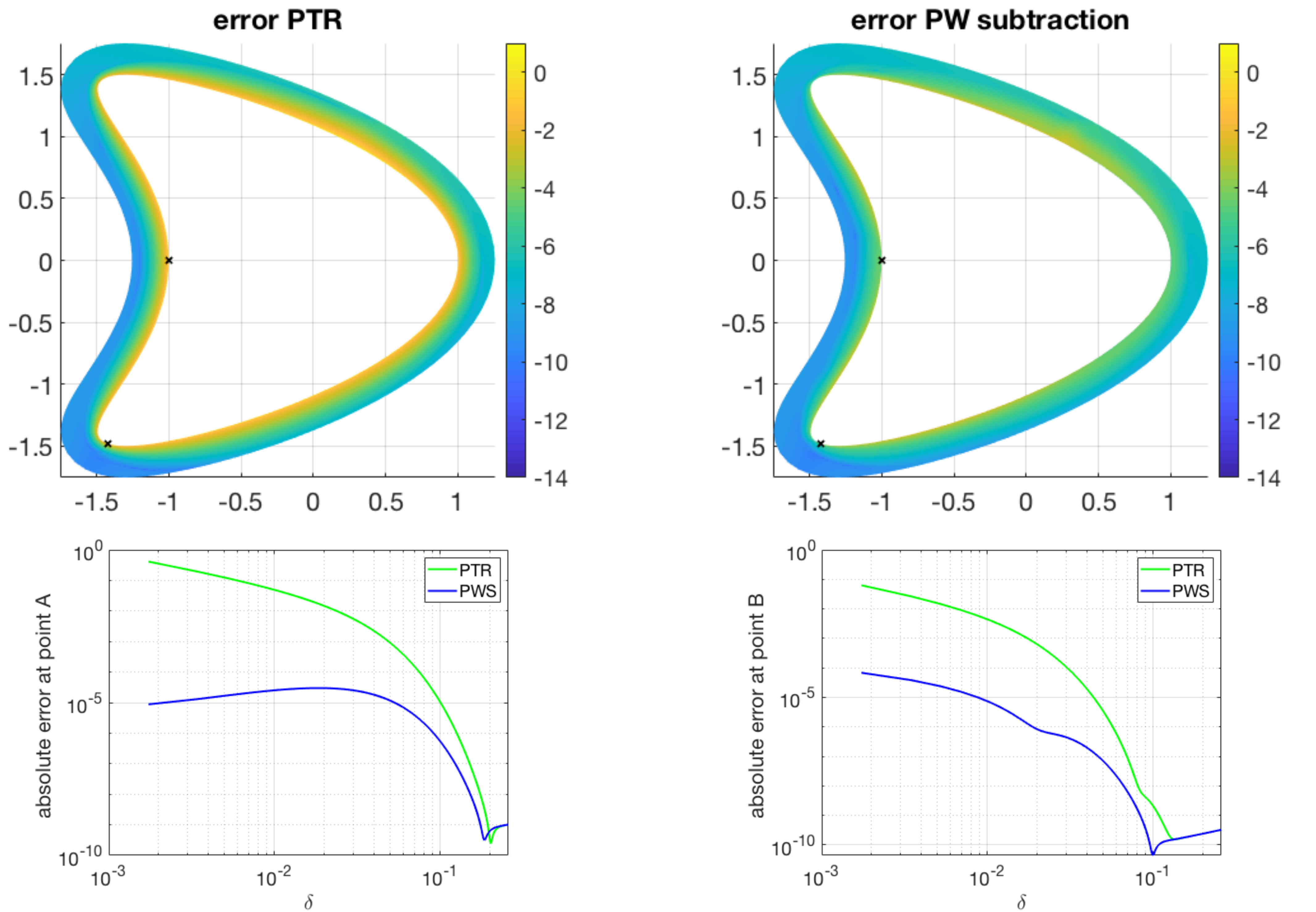




Subtraction technique for Helmholtz

Test using Periodic Trapezoid Rule (PTR) with N = 128 for $u(x) := \frac{i}{4} H_0^{(1)}(k|x - x_0|)$ $k = 5$



Outline

- Introduction
- Subtraction techniques for Laplace's equation
- Extension to Helmholtz
- Conclusion

Summary

Due to sharply peaked behavior of layer potentials' kernel, one makes an O(1) error for close evaluation.

Subtraction techniques help reduce the error (for free)

2D Helmholtz and Laplace problems

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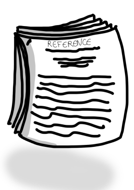
Other techniques:

Kernel/singularity subtraction techniques

Asymptotic approximations



Perez-Arancibia (2018)

Carvalho, Khatri, Kim (2020)

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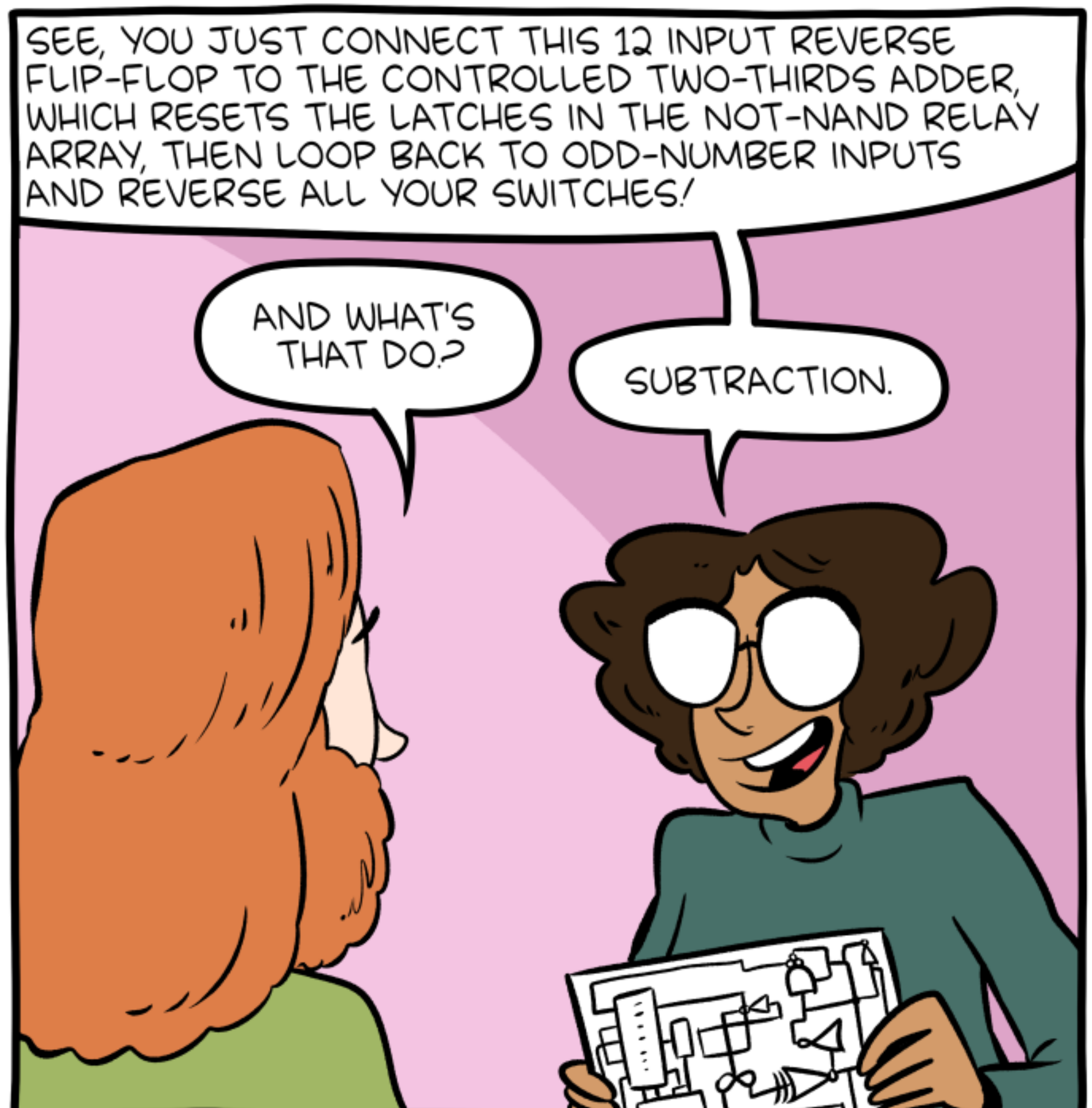
Perspectives:

Stokes flow (3D)

Scattering problem in plasmonics (transmission problem)

Thank you for your attention.



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