Math 23 Calculus III

Unit 2 Exam - Solutions

1. $$f(x,y,z) = xe^y \sin z$$

(a) (i) $\vec{\nabla} f(0,0,\frac{\pi}{2})$ is a vector pointing in the direction in which f at $(0,0,\frac{\pi}{2})$ is increasing fastest. Now,

$$\vec{\nabla} f(x,y,z) = e^y \sin z\,\vec{i} + xe^y \sin z\,\vec{j} + xe^y \cos z\,\vec{k}$$

and so

$$\vec{\nabla} f(0,0,\tfrac{\pi}{2}) = \vec{i}$$

(ii) $-\vec{\nabla} f(0,0,\pi/2)$ is a vector pointing in the direction in which f at $(0,0,\pi/2)$ is decreasing fastest. I.e. $-\vec{i}$.

(b) Required vector $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ satisfies

$$\vec{\nabla} f(0,0,\tfrac{\pi}{2}) \cdot \vec{v} = 0$$

i.e. $$\vec{i} \cdot (v_1\vec{i} + v_2\vec{j} + v_3\vec{k}) = 0$$

i.e. $$v_1 = 0$$

Take $\vec{v} = \vec{j}$.

2. $$f(x,y,z) = xy + z^2$$

$$\vec{\nabla} f(x,y,z) = y\,\vec{i} + x\,\vec{j} + 2z\,\vec{k}$$

A unit vector in the direction of $\vec{i} + 2\vec{j} + 3\vec{k}$ is

$$\vec{u} = \frac{1}{\sqrt{14}}\vec{i} + \frac{2}{\sqrt{14}}\vec{j} + \frac{3}{\sqrt{14}}\vec{k}.$$

Hence,

$$D_{\vec{u}} f(1,1,1) = \vec{\nabla} f(1,1,1) \cdot \vec{u}$$
$$= (\vec{i} + \vec{j} + 2\vec{k}) \cdot \left(\frac{1}{\sqrt{14}}\vec{i} + \frac{2}{\sqrt{14}}\vec{j} + \frac{3}{\sqrt{14}}\vec{k}\right)$$

$$= \frac{1}{\sqrt{14}} + \frac{2}{\sqrt{14}} + \frac{6}{\sqrt{14}}$$

$$= \frac{9}{\sqrt{14}}.$$

3.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= -\sin(x^2+y^2)\cdot 2x\cdot\cos v + (-\sin(x^2+y^2)\cdot 2y)\cdot\sin v$$

$$= -2u\sin^2 u$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= -\sin(x^2+y^2)\cdot 2x\cdot(-u\sin v) + (-\sin(x^2+y^2)\cdot 2y)\cdot u\cos$$

$$= 0.$$

4.

$$f(x,y) = \ln(1+x^2-y)$$

$$f(0,0) = 0$$

$$f_x(x,y) = \frac{2x}{1+x^2-y} \quad ; \quad f_x(0,0) = 0$$

$$f_y(x,y) = -\frac{1}{1+x^2-y} \quad ; \quad f_y(0,0) = -1.$$

$$f_{xx}(x,y) = \frac{2\cdot(1+x^2-y) - 2x\cdot 2x}{(1+x^2-y)^2}$$

$$= \frac{2-2x^2-2y}{(1+x^2-y)^2} \quad ; \quad f_{xx}(0,0) = 2$$

$$f_{xy}(x,y) = 2x(-1)(1+x^2-y)^{-2}\cdot(-1)$$

$$= \frac{2x}{(1+x^2-y)^2} \quad ; \quad f_{xy}(0,0) = 0.$$

$$f_{yy}(x,y) = -(-1)(1+x^2-y)^{-2}\cdot(-1)$$
$$= -\frac{1}{(1+x^2-y)^2} \quad ; \quad f_{yy}(0,0) = -1.$$

Hence,

$$Q(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{f_{xx}(0,0)}{2}x^2 + f_{xy}(0,0)xy + \frac{f_{yy}(0,0)}{2}y^2$$
$$= 0 + 0\cdot x + (-1)y + \frac{2}{2}x^2 + 0\cdot xy + \frac{(-1)}{2}y^2$$
$$= -y + x^2 - \frac{1}{2}y^2.$$

5.(a) For $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{\partial}{\partial x}\frac{xy}{\sqrt{x^2+y^2}}$$
$$= \frac{y\cdot\sqrt{x^2+y^2} - xy\cdot\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x}{x^2+y^2}$$
$$= \frac{y^3}{(x^2+y^2)^{3/2}}$$

and

$$f_y(x,y) = \frac{\partial}{\partial y}\frac{xy}{\sqrt{x^2+y^2}}$$
$$= \frac{x\sqrt{x^2+y^2} - xy\cdot\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y}{x^2+y^2}$$
$$= \frac{x^3}{(x^2+y^2)^{3/2}}$$

Hence f_x and f_y exist and are continuous for $(x,y) \neq (0,0)$ and so f is differentiable for $(x,y) \neq (0,0)$.

(b) Define a function g by

$$g(t) = \begin{cases} f(t,t) & t \neq 0 \\ 0 & t = 0. \end{cases}$$

If f is differentiable at $(0,0)$, then g is differentiable at 0. Now,

$$g(t) = \frac{t \cdot t}{\sqrt{t^2+t^2}} = \frac{1}{\sqrt{2}}|t| \quad \text{for all } t.$$

Since g isn't differentiable at 0, f isn't differential at $(0,0)$.

6. (a) We have that

$$xyz = 32$$

$$\text{or} \quad z = \frac{32}{xy}.$$

Now,

$$S = 2xy + 2xz + 2yz$$

and so

$$S(x,y) = 2xy + \frac{64}{x} + \frac{64}{y}$$

(b) Domain of S is $\{(x,y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$.

(c).

$$S_x(x,y) = 2y - \frac{64}{x^2}$$

and

$$S_y(x,y) = 2x - \frac{64}{y^2}$$

Critical points in the domain satisfy $S_x = S_y = 0$.

That is,

$$2y - \frac{64}{x^2} = 0$$

$$2x - \frac{64}{y^2} = 0$$

That is,

$$2x^2 y = 64 \quad \text{—— } 1$$
$$2x y^2 = 64 \quad \text{—— } 2$$

Hence, $2x^2y = 2xy^2$ or $x = y$.
Substituting into 1 gives $2x^3 = 64$ or $x = 32^{1/3}$.
Hence $(32^{1/3}, 32^{1/3})$ is the only critical point of S. Now,

$$S_{xx}(x,y) = \frac{128}{x^3}, \quad S_{xy}(x,y) = 2 \;\&\; S_{yy}(x,y) = \frac{128}{y^3}$$

and so

$$D(32^{1/3}, 32^{1/3}) = S_{xx}(32^{1/3}, 32^{1/3})S_{yy}(32^{1/3}, 32^{1/3}) - S_{xy}^2(32^{1/3}, 32^{1/3})$$

$$= \frac{128}{32} \cdot \frac{128}{32} - 2^2$$

$$= 12 > 0$$

Since $S_{xx}(32^{1/3}, 32^{1/3}) = 4 > 0$, the critical point is a local minimum.

(d) We have that $S(32^{1/3}, 32^{1/3}) = 6 \cdot 32^{2/3}$.

If $0 < x < \frac{1}{3}$, then $S(x,y) > \frac{64}{x} > 6 \cdot 32 > 6 \cdot 32^{2/3}$

If $0 < y < \frac{1}{3}$, then $S(x,y) > \frac{64}{y} > 6 \cdot 32 > 6 \cdot 32^{2/3}$

If $x > 288$ and $y \geq \frac{1}{3}$, then

$$S(x,y) > 2xy > 6 \cdot 32 > 6 \cdot 32^{2/3}$$

If $x \geq \frac{1}{3}$ and $y > 288$, then

$$S(x,y) > 2xy > 6 \cdot 32 > 6 \cdot 32^{2/3}$$

(e) By the extreme value theorem, we know that S has a global minimum, say (x_0, y_0), on R. That is,

$$S(x_0, y_0) \leq S(x,y) \quad \text{for all } (x,y) \text{ in } R.$$

In particular, we have that $S(x_0, y_0) \leq S(32^{1/3}, 32^{1/3})$. By part (d),

$$S(x,y) > S(32^{1/3}, 32^{1/3}) \geq S(x_0, y_0)$$

for all points (x,y) outside of R. Hence, (x_0, y_0) is a global minimum for S. Since this isn't a boundary point for S, it must be a critical point of S. But $(32^{1/3}, 32^{1/3})$ is the only critical point of S and so it must be the global minimum of S.