MATH 24 – EXAM 3 – SOLUTIONS

1. Consider the equation
$$y'' + \omega_0^2 y = \cos\omega_1 t\ .$$

   (a) Find the general solution of the homogeneous equation.

   (b) Find the particular solution of the non-homogeneous equation using the method of <u>Undetermined Coefficients</u> assuming that $\omega_1 \neq \omega_0$.

   (c) What is the suitable guess for the particular solution if $\omega_1 = \omega_0$?

   (d) Suppose $\omega_1 > \omega_0$, but as time evolves, ω_1 is getting closer and closer to ω_0. What happens to the qualitative behavior of the solution in this process? Is this change of behavior consistent with your answer to part (c)?

SOLUTION

(a) Making the guess $y = e^{rt}$ gives $r^2 + \omega_0^2 = 0$ and $r_{1,2} = \pm i\omega_0$. Hence $y_h = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$.

(b) Since $\omega_1 \neq \omega_0$, this is a regular case for UC. Hence the guess is $y_p = A\cos\omega_1 t + B\sin\omega_1 t$. Substituting this guess in the equation gives
$$(-\omega_1^2 + \omega_0^2)(A\cos\omega_1 t + B\sin\omega_1 t) = \cos\omega_1 t\ .$$
The solution is $B = 0$ and $A = \dfrac{1}{\omega_0^2 - \omega_1^2}$, so $y_p = \dfrac{\cos\omega_1 t}{\omega_0^2 - \omega_1^2}$.

(c) If $\omega_1 = \omega_0$ this is an exceptional case for UC. Therefore, the suitable guess is multiplied by t as $y_p = At\cos\omega_1 t + Bt\sin\omega_1 t$.

(d) As ω_1 approaches ω_0, the y_p in part (b) has oscillations that grow without bound. When $\omega_1 = \omega_0$, i.e., part (c), this is the case of resonance, where the additional factor t corresponds to the unbounded growth of the amplitude. Hence, the unbounded growth of $A(\omega_1)$ is the precursor of resonance.

Note: a more complete explanation of the limit as ω_1 approaches ω_0 requires incorporating the initial data with the general solution (see lecture on April 10).

2. Consider the equation

$$y'' - 2y' + y = \frac{e^t}{t} \ .$$

(a) Find the general solution of the homogeneous equation.

(b) Find the Wronskian of the fundamental solutions of the homogeneous equation.

(c) Find the particular solution of the non-homogeneous equation using the method of <u>Variation of Parameters</u>.

(d) Convert this equation to a system of first-order equations and write it using vector-matrix notation.

SOLUTION

(a) Making the guess $y = e^{rt}$ gives $r^2 - 2r + 1 = 0$ and $r_{1,2} = 1$. Hence $y_h = c_1 e^t + c_2 t e^t$.

(b) Setting $y_1 = e^t$ and $y_2 = te^t$ gives $W = y_1 y_2' - y_1' y_2 = e^{2t}$.

(c) Using VOP, $y_p = v_1 y_1 + v_2 y_2$, $f(t) = e^t/t$, and

$$\begin{aligned} v_1' &= -\frac{y_2 f}{W} = -\frac{te^t \, e^t/t}{e^{2t}} = -1 \ , \\ v_2' &= \frac{y_2 f}{W} = \frac{e^t \, e^t/t}{e^{2t}} = \frac{1}{t} \ . \end{aligned}$$

Therefore, $v_1 = -t, v_2 = \ln t$, and $y_p = te^t(\ln t - 1)$. Note that another particular solution is just $y_p = te^t \ln t$, since the te^t part is already contained in the homogeneous solution.

(d) Setting $x_1 = y$ and $x_2 = y'$ gives the 2x2 system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, with $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{b}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{e^t}{t}$.

3. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues.

(b) Find the eigenvectors.

(c) What is the dimension of the span of all the eigenvectors?

(d) Does the algebraic system $A\mathbf{x} = \mathbf{0}$ have a unique solution?

SOLUTION

(a) Since A is upper-triangular, $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 2$.

(b) For $\lambda_1 = 0$ the augmented form for $(A - \lambda_1 I)\mathbf{v} = \mathbf{0}$ is

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right],$$

which gives the eigenvector (up to scalar multiplication) $\mathbf{v_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda_2 = \lambda_3 = 2$ the augmented form is

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which is a deficient case with only one eigenvector, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(c) The span of $\mathbf{v_1}$ and $\mathbf{v_2}$ is a 2-dimensional subspace of $\mathbb{R}^3$, i.e., a plane.

(d) Since $\lambda_1 = 0$, it follows that $det(A) = 0$ and, therefore, $A\mathbf{x} = \mathbf{0}$ does not have a unique solution.

4. Determine whether the following statements are TRUE or FALSE. Note: you must write the entire word TRUE or FALSE. You do not need to show your work for this problem.

(a) Let $y'' - 2y' + y = te^t \cos t$. Then $y_p = e^t(A\cos t + B\sin t)$ is a suitable guess for the particular solution.

(b) Let $y'' + y' = t^2 + 1$. Then $y_p = At^2 + Bt + C$ is a suitable guess for the particular solution.

(c) Suppose the eigenvalues of a matrix are $\{-1, 0, 1\}$. Then the matrix is invertible.

(d) Let
$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 7 \\ -1 & 0 & 2 \end{bmatrix}.$$
Then the sum of the eigenvalues of is 3.

(e) The eigenvalues of the fourth-order equation $y'''' - y = 0$ are distinct (i.e., different from each other).

SOLUTION

(a) TRUE. This is not an exceptional case, since the characteristic roots are $r_{1,2} = 1$ and the corresponding homogeneous solutions are e^t and te^t, whereas, the forcing function, $e^t \cos t$, corresponds to $r = 1 \pm i$. Therefore, the standard guess will work, which is the given y_p. If you don't believe it, you can check that the solution is $y_p = e^t(-t\cos t + 2\sin t)$.

(b) FALSE. This is an exceptional case, since the polynomial on the right-hand side shares the underlying solution with one of the roots, $r_1 = 0$. Hence, the right guess is $y_p = At^3 + Bt^2 + Ct$.

(c) FALSE. Since $\lambda_2 = 0$, it follows that $det(A) = 0$ and A is not-invertible.

(d) TRUE. The sum of the eigenvalues is equal to $Tr(A) = 3$.

(e) TRUE. Setting $y = e^{rt}$ gives $r^4 - 1 = 0$. Taking the square-root gives $r^2 = \pm 1$ and another square-root yields $r_{1,2} = \pm 1$, $r_{3,4} = \pm i$. Hence, the roots are distinct.

5. Consider the system of differential equations $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}$.

(a) Find the eigenvalues.

(b) Find the eigenvectors.

(c) Find the general solution.

(d) Solve the initial value problem with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(e) What is the stability structure of the equilibrium solution?

SOLUTION

(a) Using the formula $\lambda^2 - Tr(A)\lambda + |A| = 0$ gives $\lambda^2 + 1 = 0$. Therefore, $\lambda_{1,2} = \pm i$ (i.e., $\alpha = 0$, $\beta = 1$).

(b) $(A - \lambda_1 I)\mathbf{v} = \mathbf{0}$ gives

$$\left[\begin{array}{cc|c} -3-i & 2 & 0 \\ -5 & 3-i & 0 \end{array}\right],$$

which gives one eigenvector as $\mathbf{v_1} = \begin{bmatrix} 2 \\ 3+i \end{bmatrix}$ and the other one (corresponding to λ_2) is $\mathbf{v_2} = \mathbf{v_1}^* = \begin{bmatrix} 2 \\ 3-i \end{bmatrix}$.

(c) Since this is a complex case we decompose the first eigenvector to real and imaginary parts as

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ 3+i \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv \mathbf{p} + i\mathbf{q}$$

and use the formulae for the two eigensolutions, to obtain

$$\begin{aligned} \mathbf{x_1}(t) &= \mathbf{p}\cos t - \mathbf{q}\sin t = \begin{bmatrix} 2\cos t \\ 3\cos t - \sin t \end{bmatrix}, \\ \mathbf{x_2}(t) &= \mathbf{q}\cos t + \mathbf{p}\sin t = \begin{bmatrix} 2\sin t \\ \cos t + 3\sin t \end{bmatrix}. \end{aligned}$$

With these, the general solution is given by $\mathbf{x}(t) = c_1\mathbf{x_1} + c_2\mathbf{x_2}$.

(d) Setting $t = 0$ in the general solution (i.e., $\cos t = 1$ and $\sin t = 0$) and equating to the initial values gives

$$\mathbf{x}(0) = \begin{bmatrix} 2c_1 \\ 3c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

and $c_1 = 1$ and $c_2 = 0$. Therefore, $\mathbf{x}(t) = \mathbf{x_1} = \begin{bmatrix} 2\cos t \\ 3\cos t - \sin t \end{bmatrix}$.

THE END