FINAL EXAM, MATH 24, FALL 2008

Instructions: Do not begin the exam until you are instructed to do so. You may write on the exam sheet, but ONLY what is written in your bluebook will be graded.

For each problem, you must show all work in order to receive credit. Partial credit will be given when appropriate, even if the final answer is not correct, but an answer with no work shown will receive zero credit regardless of correctness. You may not use any text, notes, or calculators on this exam, and collaboration is not allowed.

1. (10 points) Solve the IVP

$$\begin{aligned} y' + ty &= t \\ y(0) &= 3 \end{aligned}$$

Using the integrating factor method (but variation of parameters and separation of variables are also appropriate), we have

$$\mu = e^{\int t\ dt} = e^{\frac{1}{2}t^2}$$

and then

$$\begin{aligned} y'e^{\frac{1}{2}t^2} + te^{\frac{1}{2}t^2}y &= te^{\frac{1}{2}t^2} \\ &\Rightarrow [ye^{\frac{1}{2}t^2}]' = te^{\frac{1}{2}t^2} \\ &\Rightarrow ye^{\frac{1}{2}t^2} = \int te^{\frac{1}{2}t^2}\ dt \\ &= \int e^u\ du = e^u + C = e^{\frac{1}{2}t^2} + C \\ &\Rightarrow y = 1 + Ce^{-\frac{1}{2}t^2} \\ y(0) &= 1 + C = 3 \Rightarrow C = 2 \\ &\Rightarrow \boxed{y(t) = 1 + 2e^{-\frac{1}{2}t^2}} \end{aligned}$$

2. For the inhomogeneous system

$$\begin{aligned} x' &= 4x - y + 6 \\ y' &= 2x + y \end{aligned}$$

a. (10 points) Find the equilibrium solution(s).

We find v-nullclines by setting x' equal to zero, and h-nullclines by setting y' equal to zero. Thus, the v-nullcline is the line $y = 4x + 6$ and the h-nullcline is the line $y = -2x$. Equilibrium solutions occur at the intersection of v- and h-nullclines, so setting the right sides of these equations equal to each other, we have

$$4x + 6 = -2x$$

which shows that the x-coordinate of the intersection is -1. Plugging this value in to either line equation shows that the y-coordinate is 2. Thus the equilibrium solution is (-1,2).

b. (5 points) Write the system in matrix-vector form.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

3. (10 points) What value or values of k make the vectors $\begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} k \\ -1 \\ 3 \end{bmatrix}$ linearly dependent?

These vectors are linearly dependent if the matrix for which these are column vectors is singular. Thus, we need to find values of k such that $\begin{vmatrix} 1 & 1 & k \\ 2 & 0 & -1 \\ k & 0 & 3 \end{vmatrix} = 0$. Expanding about the second column, we have

$$-\begin{vmatrix} 2 & -1 \\ k & 3 \end{vmatrix} = -(6+k) = 0$$

So $\boxed{k = -6}$ is the unique value such that these vectors are linearly dependent. Finding nonzero values c_1, c_2, and c_3 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} k \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

yields a similar analysis.

4. For the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$,

a. (10 points) Find the eigenvalues.

$$\begin{aligned} \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} &= \lambda^2 - 4\lambda + 4 = 0 \\ &\Rightarrow (\lambda - 2)^2 = 0 \\ &\Rightarrow \lambda = 2 \text{ (repeated)} \end{aligned}$$

b. (10 points) Find two linearly independent eigenvectors.

$$\begin{aligned} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] &\Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ &\Rightarrow x_2 = r,\ x_1 = r \end{aligned}$$

Choosing $r = 1$, we have $\boxed{\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$.

Since the basis for the solution space of $(A - \lambda I)\mathbf{v}_1 = \mathbf{0}$ had dimension 1 (instead of 2), we must find a generalized eigenvector $\mathbf{u}$ such that $(A - \lambda I)\mathbf{u} = \mathbf{v}_1$.

$$\begin{aligned} \left[\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] &\Rightarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \end{array}\right] \\ &\Rightarrow y_2 = s,\ y_1 = s - 1 \end{aligned}$$

Thus, we have $\mathbf{u} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Thus, a *generalized eigenvector* for this matrix is given by this formula for any value of the free parameter s. Choosing $s = 0$, a generalized eigenvector is $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

5. (15 points) Find the general solution to the differential equation

$$y'' - 2y' - 2y = 2t$$

Using characteristic roots to find the homogeneous solution, we have

$$\begin{aligned} r^2 - 2r - 2 &= 0 \\ &\Rightarrow r_{1,2} = \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3} \\ &\Rightarrow y_h = c_1 e^{(1+\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t} \end{aligned}$$

Using undetermined coefficients to find the particular solution, and noting that the forcing function $f(t) = 2t$ does NOT solve the homogeneous equation, we have

$$\begin{aligned} y_p &= At + B \\ y_p' &= A \\ y_p'' &= 0 \\ &\Rightarrow -2A - 2At - 2B = 2t \end{aligned}$$

which yields $A = -1$, $B = 1$, so $y_p = 1 - t$, and $\boxed{y(t) = 1 - t + c_1 e^{(1+\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t}}$

6. (2 points) Find the Laplace Transform for the function $f(t) = \sinh(t)$, $\underline{\underline{\text{for } s > 1}}$. HINT: Express the function in terms of exponentials.

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \sinh(t)\, dt = \int_0^\infty \frac{1}{2}(e^t - e^{-t}) e^{-st}\, dt \\ &= \frac{1}{2} \int_0^\infty [e^{(1-s)t} - e^{-(1+s)t}]\, dt \\ &= \frac{1}{2} \left[\frac{1}{1-s} e^{(1-s)t} - \frac{-1}{(1+s)} e^{-(1+s)t} \right]_{t=0}^{\infty} \\ &= \frac{1}{2} \left[\frac{1}{1-s} e^{(1-s)t} + \frac{1}{(1+s)} e^{-(1+s)t} \right]_{t=0}^{\infty} \\ &= \frac{1}{2} \left[(0+0) - \left(\frac{1}{1-s} + \frac{1}{1+s} \right) \right] \\ &= \frac{1}{2} \left(\frac{1}{s-1} - \frac{1}{s+1} \right) \\ &= \frac{1}{2} \frac{(s+1) - (s-1)}{s^2 - 1} \\ &= \frac{1}{2} \frac{2}{s^2 - 1} \\ &\Rightarrow \boxed{F(s) = \tfrac{1}{s^2-1}} \end{aligned}$$

7. The following statements are incorrect, or at least not *entirely* accurate. Explain why with complete sentences and/or an example.

a. (3 points) The column vectors of any square matrix form a basis for the column space of that matrix.

The column vectors of a matrix always *span* the column space of that matrix, but they only form a *basis* for the column space if the column vectors are linearly independent. Recall that vectors in *any* basis MUST be linearly independent. Thus, this is true for *non-singular* matrices only. Also, the matrix need not be square, but if it isn't, the number of rows must be greater than the number of columns.

b. (3 points) Any plane in the vector space $\mathbb{R}^3$ can be specifically considered as the vector space $\mathbb{R}^2$, so all planes in $\mathbb{R}^3$ are vector subspaces.

Not all planes in $\mathbb{R}^3$ are vector subspaces of $\mathbb{R}^3$. Recall that a vector space must contain the zero vector, so only planes that pass through the origin can be vector subspaces of $\mathbb{R}^3$. Furthermore, *only* the plane $z = 0$ (i.e. the xy-plane) in $\mathbb{R}^3$ is the specific vector subspace $\mathbb{R}^2$.

c. (3 points) The linear system $A\mathbf{x} = \mathbf{b}$ will have infinitely many solutions if A is a singular matrix.

This is true if the system is consistent. But it is possible that the system is inconsistent, and then no solutions would exist. If the matrix is singular, all we know is that there are EITHER no solutions or infinitely many solutions.

d. (3 points) A repeated characteristic root of a second-order differential equation can be either real or complex.

The characteristic roots of a second-order DE are the roots of a quadratic polynomial. These roots can be found by the quadratic formula. If the root is repeated, then the discriminant is zero, and therefore the root must be real, since only a negative discriminant leads to complex roots (conversely, if the discriminant is negative, leading to complex roots, then, due to the $\pm$ in the numerator of the quadratic formula, the roots must be distinct, and thus a repeated root cannot be obtained). It is possible in general to have repeated complex roots, but only for higher-order DE's. If a 2nd-order DE has a repeated root, it must be real.

e. (3 points) If a 2nd-order linear system of (first-order) DE's has a repeated eigenvalue, then the eigenvectors cannot possibly span the eigenspace, and you must use a generalized eigenvector (i.e. $\mathbf{u}$ such that $(A - \lambda I)\mathbf{u} = \mathbf{v}$, where λ is an eigenvalue, and $\mathbf{v}$ is its eigenvector) to obtain 2 linearly independent eigenvectors.

If you obtain two free parameters in the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$, you can get two linearly independent eigenvectors from the repeated eigenvalue. Consider the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The repeated eigenvalue is 1, and thus the system becomes

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and both components are arbitrary, leading to $r \begin{bmatrix} 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and thus a basis for the eigenspace of the repeated eigenvalue is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, which has dimension 2. Note that we were not required to use the generalized eigenvector **u.** The generalized eigenvector would have been required only if there had been one free parameter instead of two.

f. (3 points) If a matrix transforms a vector such that the resulting vector remains in the same vector space as the original, then the original vector is an eigenvector of the transformation matrix.

If a matrix transforms a vector such that the resulting vector is PARALLEL (or anti-parallel) to the original, then the vector is an eigenvector of the transformation matrix. Being in the same vector space is not sufficient.

8. A tank contains 100 gallons of pure water. At time $t = 0$, saltwater containing 5 pounds of salt per gallon is pumped in at a rate of 2 gallons per minute. The mixture is drained such that the volume of liquid in the tank remains constant.

a. (10 points) Find an expression for the amount of salt in the tank at time t.

$$\begin{aligned} Q' &= \text{(concentration in)(flow rate in)} - \text{(concentration out)(flow rate out)} \\ &= (5\frac{lbs}{gal})(2\frac{gal}{\min}) - (\frac{Q\ lbs}{100\ gal})(2\frac{gal}{\min}) \\ \Rightarrow &\ Q' = 10 - \frac{Q}{50} \end{aligned}$$

From inspection of the DE, we see that a particular solution is $Q_p = 500$, and that the homogeneous solution is a (decaying) exponential $Q_h = Ce^{-\frac{t}{50}}$. Thus, the general solution is $Q(t) = 500 + Ce^{-\frac{t}{50}}$. Using the initial condition $Q(0) = 0$, we see that $C = -500$, and thus the amount of salt in the tank at time t is

$$\boxed{Q(t) = 500(1 - e^{-\frac{t}{50}})}$$

b. (5 points) As $t \to \infty$, how much salt is in the tank?

As $t \to \infty$, the exponential approaches zero, and the amount of salt in the tank approaches 500 pounds. Another way to see this is that as $t \to \infty$, the concentration in the tank approaches the concentration of the incoming saltwater. This saltwater has a concentration of $5\frac{lbs}{gal}$, and since the volume of the tank is held constant at 100 gallons, the amount of salt is again 500 pounds.