Math 24 Spring 2009 - Exam 1 Solutions

Prepared by Vikram Rao

Lecturer: Boaz Ilan

1. (30 points) Consider the equation $\frac{dy}{dt} = ty + ty^2$.

   (a) Define what an equilibrium solution is <u>in general</u>.

   (b) Find the equilibrium solutions of the equation above.

   (c) Sketch the direction field and determine the stability of the equilibrium solutions.

   (d) Find the general solution. What is its long-time behavior?

   (e) Perform the transformation $v(t) = \frac{1}{y(t)}$, obtain a differential equation for $v(t)$, and classify this equation.

SOLUTION:

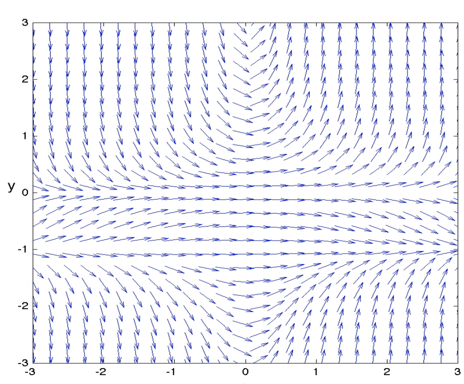
(a) An equilibrium solution is a constant solution, that is $y(t) = c$, where c is a constant. Therefore, $\frac{dy}{dt} = y' = 0$. Its graph the (t, y) plane is a horizontal line.

(b) Equilibrium solutions can be found by setting $f(t, y) = 0$. Here

$$\frac{dy}{dt} = ty + ty^2 = 0 \Rightarrow ty(y+1) = 0$$

$\therefore y = 0$ and $y = -1$ are equilibrium solutions. Note that $t = 0$ is ***not*** an equilibrium solution or a solution at all.

(c) The direction field has been plotted below.



Based on the results below, $y = 0$ is an unstable equilibrium solution and

$y = -1$ is a stable equilibrium solution.

(d) Using separation of variables

$$\frac{dy}{dt} = ty(y+1) \Rightarrow \frac{dy}{y(y+1)} = t\,dt \Rightarrow \int \frac{dy}{y(y+1)} = \int t\,dt \Rightarrow \int \frac{dy}{y(y+1)} = \frac{t^2}{2} + C$$

For $\int \frac{dy}{y(y+1)}$, we can use partial fraction decomposition, $\int \frac{dy}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$

$$\Rightarrow 1 = A(y+1) + B(y) \Rightarrow \therefore A = 1, B = -1, \Rightarrow \int \frac{dy}{y(y+1)} = \ln\left|\frac{y}{y+1}\right|$$

$$\Rightarrow \ln\left|\frac{y}{y+1}\right| = \frac{t^2}{2} + C \Rightarrow y = \frac{Ke^{\frac{t^2}{2}}}{1 - Ke^{\frac{t^2}{2}}} \; ; \; (K = e^C)$$

For the long-time behavior $(t \to \infty)$, using L'Hospital's Rule once, y would go to -1. This should not be surprising because $y = -1$ is the stable equilibrium solution.

(e) $v = \frac{1}{y} \Rightarrow y = \frac{1}{v} \Rightarrow y' = -\frac{1}{v^2}v' \Rightarrow -\frac{1}{v^2}v' = \frac{t}{v} + \frac{t}{v^2} \Rightarrow v' = -vt - t$

This equation is 1st order, linear, nonhomogenous, variable coefficients, separable, and nonautonomous.

2. (30 points) Consider the equation $\frac{dy}{dt} + \frac{y}{\sqrt{t}} = e^{-\sqrt{t}}$

(a) Classify this equation as best as you can.

(b) Explain what Picard's theorem says about the existence and uniqueness of a solution with the initial value $y(0) = 1$.

(c) Find the general solution.

(d) Find the solution with initial value $y(0) = 1$. Is your answer consistent with part (b)?

SOLUTION:

(a) 1st order, linear, nonhomogeneous, variable coefficients, nonseparable, nonautonomous

(b) Picard's theorem is <u>inconclusive</u> about existence and uniqueness because $f(t, y) = e^{-\sqrt{t}} - \frac{y}{\sqrt{t}}$ is undefined at $(t, y) = (0,1)$.

(c) We need to use the Integrating Factor method so first find $\mu(t) = e^{\int t^{-1/2}} = e^{2\sqrt{t}}$. Next, we will multiply both sides by $\mu(t)$. The left hand side is a perfect derivative and we get that

$$\left(e^{2\sqrt{t}}y\right)' = e^{\sqrt{t}} \Rightarrow \int \left(e^{2\sqrt{t}}y\right)' = \int e^{\sqrt{t}}dt$$

To integrate $\int e^{\sqrt{t}}dt$ we use integration by parts, but first make a substitution of $u = \sqrt{t}$. Therefore, the integral of the right-hand side of the equation above is

$$u^2 = t \Rightarrow 2u\ du = dt \Rightarrow 2\int ue^u\ du = 2ue^u - 2e^u \Rightarrow 2\sqrt{t}e^{\sqrt{t}} - 2e^{\sqrt{t}}$$

Using this integral and the equation we obtain the general solution as

$$\left(e^{2\sqrt{t}}y\right) = 2\sqrt{t}e^{\sqrt{t}} - 2e^{\sqrt{t}} + C \Rightarrow y = \frac{2\sqrt{t}}{e^{\sqrt{t}}} - \frac{2}{e^{\sqrt{t}}} + \frac{C}{e^{2\sqrt{t}}}$$

(d) Plugging in $t = 0$ and $y = 1$, we get that $C = 3$. Therefore,

$y = \frac{2\sqrt{t}}{e^{\sqrt{t}}} - \frac{2}{e^{\sqrt{t}}} + \frac{3}{e^{2\sqrt{t}}}$. This is consistent with part (b) because Picard's theorem was inconclusive. In other words, any answer would have been consistent with Picard's theorem.

3. (20 points) Answer the following TRUE/FALSE questions. Explain your answers concisely and be sure to write the word TRUE or FALSE.

(a) $y(x) = x\sin x$ is a solution of the IVP $\dfrac{dy}{dx} - \dfrac{y}{x} = -x\cos x,\;\; y(0) = 0.$

(b) $y = t$ is an equilibrium solution of $\dfrac{dy}{dt} = t[\tan(y) - \tan(t)]$

(c) Picard's theorem guarantees existence and uniqueness of a solution to the IVP $\dfrac{dy}{dx} = \sin^{-1} y,\quad y(0) = 0.$

(d) If $y_1(t)$ and $y_2(t)$ are two solutions of the equation $\sin(t)\dfrac{dy}{dt} - ty = 0$ then $y_1 + y_2$ is also a solution of this equation.

SOLUTION:

(a) With $y(x) = x\sin x$ and $\dfrac{dy}{dx} = \sin x + x\cos x$, plugging in we get that $x\cos x = -x\cos x$, and because the left hand side is not equal to the right hand side, y is not a solution. Thus, it is FALSE.

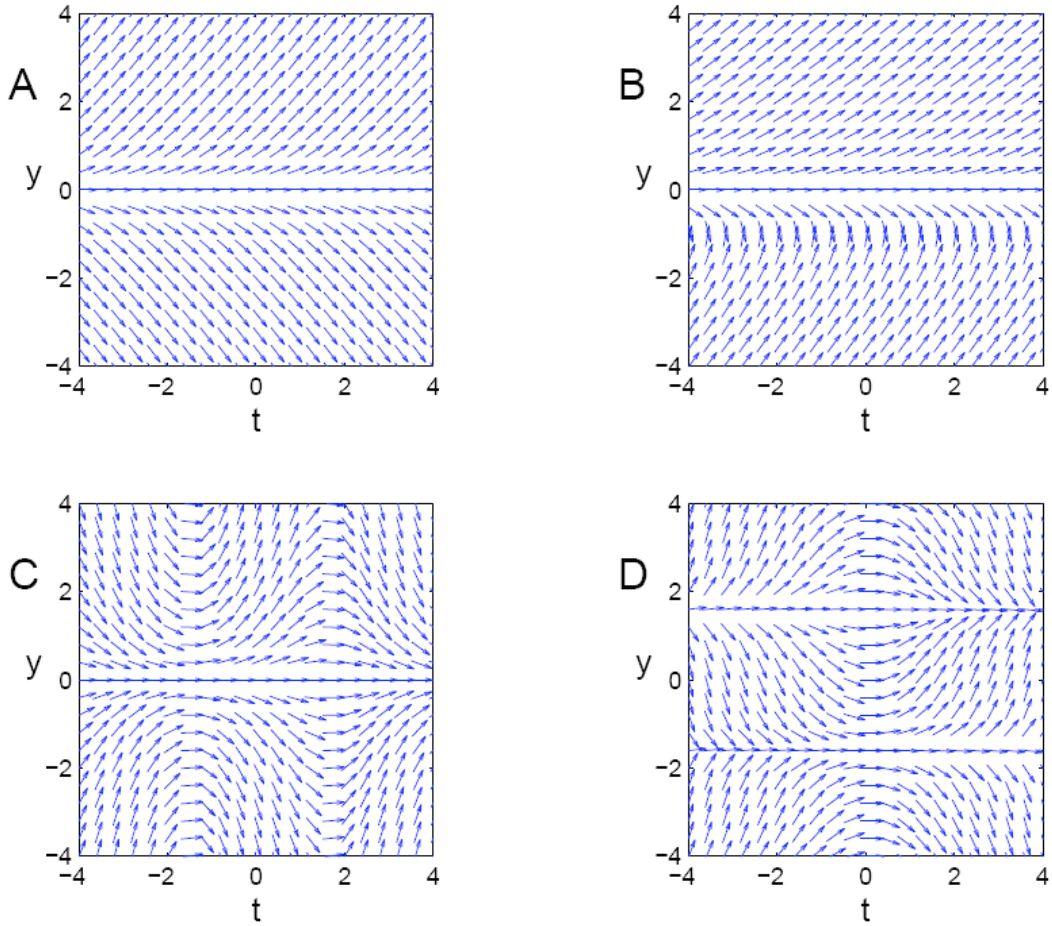
(b) Recall that from Problem 1 part (a) that an equilibrium solution has no t dependence because it is a constant solution. In this case, $y = t$ is an *isocline*, not an equilibrium solution (in fact, all equilibrium solutions are isoclines, but the converse is not true). Therefore the statement is FALSE.

(c) $f(t,y) = \dfrac{dy}{dx} = \sin^{-1} y$ is defined at $y = 0$, so it passes Picard's existence test. For uniqueness, we have $\dfrac{\partial f}{\partial y} = \dfrac{1}{\sqrt{1-y^2}}$, which is defined at $y = 0$. Hence, it passes Picard's uniqueness test and the statement is TRUE.

(d) If we know that two separate functions are solutions of the differential equation that is ***both linear and homogeneous***, their sum will also be a solution. As this differential equation is both linear and homogeneous the statement is TRUE.

4. (20 points) Match the following equations (1)-(4) with their corresponding direction fields A-D (you do not need to show your work for this problem):

$$(1)\ \frac{dy}{dt} = t\cos y,\ (2)\ \frac{dy}{dt} = y\cos t,\ (3)\ \frac{dy}{dt} = \frac{y}{y+1},\ (4)\ \frac{dy}{dt} = \tan^{-1} y,$$



ANSWERS:

(1)-D, (2)-C, (3)-B, (4)-A