

Math 24 Spring 2009 – Final Exam

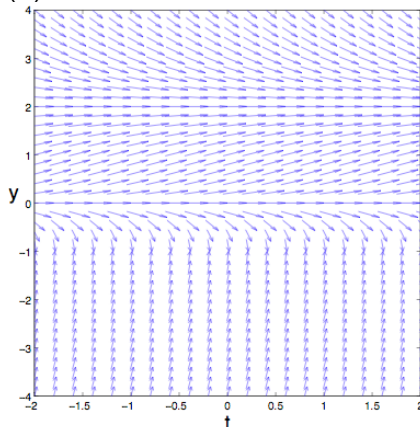
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1. Consider the differential equation (DE) $\frac{dy}{dt} = \frac{y(2-y)}{1+y}$.
- (a) Classify the equation as best you can.
 - (b) Find the equilibrium solutions.
 - (c) Using phase lines, determine the stability structure of the equilibrium solutions.
 - (d) What is the long time behavior corresponding to the initial conditions $y(0) = 1$?
 - (e) Sketch the phase lines and the solution curves corresponding to the initial conditions $y(0) = 1$ and $y(0) = 3$.

SOLUTION:

- (a) This DE is 1st order, nonlinear, autonomous, and separable
- (b) $\frac{dy}{dt} = \frac{y(2-y)}{1+y} = 0 \Rightarrow y = 0$ & $y = 2$ are the equilibrium solutions.
- (c) A direction field has been plotted below. Based upon the direction field, $y = 0$ is an unstable equilibrium solution and $y = 2$ is a stable equilibrium solution.
- (d) The solutions tend to the stable solution $y = 2$ for large t .



2. Consider the DE $\frac{dy}{dt} - \sqrt{t}y = e^{\left(\frac{2}{3}t^{\frac{3}{2}}\right)}$

(a) Find the Integrating Factor.

(b) Find the general solution.

SOLUTION:

(a) $\mu(t) = e^{\int -\sqrt{t} dt} = e^{-\frac{2}{3}t^{\frac{3}{2}}}.$

(b) Multiply both sides by the Integrating Factor. The left hand side becomes a perfect derivative of the Integrating Factor and y . Therefore,

$$\frac{d}{dt} \left[e^{-\frac{2}{3}t^{\frac{3}{2}}} y \right] = 1 \quad \Rightarrow \quad e^{-\frac{2}{3}t^{\frac{3}{2}}} y = \int 1 dt \quad \Rightarrow \quad e^{-\frac{2}{3}t^{\frac{3}{2}}} y = t + C.$$

Now we can explicitly solve for y to get

$$y = \frac{t+C}{e^{-\frac{2}{3}t^{\frac{3}{2}}}} = (t+C)e^{\frac{2}{3}t^{\frac{3}{2}}}.$$

3. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 3 \\ 0 & 4 & -1 \end{bmatrix}$

(a) What is the determinant of A ?

(b) What are the solutions of $A\mathbf{x} = 0$?

(c) What is the RREF of A ?

(d) What is the product of the eigenvalues of A ?

(e) What is the sum of the eigenvalues of A ?

SOLUTION:

(a) $|A| = 1 \begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} + 0 \begin{vmatrix} 2 & 2 \\ 0 & 4 \end{vmatrix} = -10.$

(b) Since the determinant of A is not zero, A is invertible. Therefore, $A\mathbf{x} = 0$ has only the trivial solution $\mathbf{x} = 0$.

- (c) The RREF can be found using row operations, but there is an easier method to determine the RREF of A . Since the determinant is nonzero, the RREF is the 3×3 identity matrix.
- (d) The product of the eigenvalues is the determinant of A that is -10 .
- (e) The sum of the eigenvalues is the trace of A that is 2 .
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4. Consider the DE $y'' - 2y' + y = \frac{e^t}{t}$.

- (a) Find the general solution of the homogeneous problem.
- (b) Find a particular solution of the non-homogeneous problem using the method of Variation of Parameters.
- (c) What is the general solution of the DE?

SOLUTION:

- (a) Using $y = e^{rt}$, $\Rightarrow r^2 - 2r + 1 = 0 \Rightarrow r = 1$ (double root)

$$\therefore y_h = c_1 e^t + c_2 t e^t$$

- (b) We first need to find the Wronskian. $W = \begin{vmatrix} e^t & t e^t \\ e^t & e^t + t e^t \end{vmatrix} = e^{2t}$.

$$v_1' = -\frac{y_2 f}{W} = -\frac{t e^t \left(\frac{e^t}{t} \right)}{e^{2t}} = -1 \Rightarrow v_1 = -t$$

$$v_2' = \frac{y_1 f}{W} = \frac{e^t \left(\frac{e^t}{t} \right)}{e^{2t}} = \frac{1}{t} \Rightarrow v_2 = \ln|t|$$

$$\Rightarrow y_p = y_1 v_1 + y_2 v_2 \Rightarrow y_p = -t e^t + t e^t \ln|t|$$

- (c) Using Variation of Parameters we conclude that

$$y = y_h + y_p = c_1 e^t + c_2 t e^t - t e^t + t e^t \ln|t|.$$

5. Consider the system of DEs $x_1' = 2x_1 + x_2, x_2' = x_1 - 3x_2$.

- (a) Write the system in the form $\mathbf{x}' = A\mathbf{x}$.
- (b) Find the eigenvalues of A .
- (c) Find the eigenvectors of A .
- (d) Classify the stability structure of the equilibrium solution.
- (e) Sketch the phase portrait of the solutions in the (x_1, x_2) plane.

SOLUTION:

$$(a) \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$(b) \text{ We can use that } \lambda_{1,2} = \frac{\text{Tr}(A) \pm \sqrt{(\text{Tr}(A))^2 - 4 \det(A)}}{2} = \frac{-1 \pm \sqrt{29}}{2} = \frac{-1}{2} \pm \frac{\sqrt{29}}{2}$$

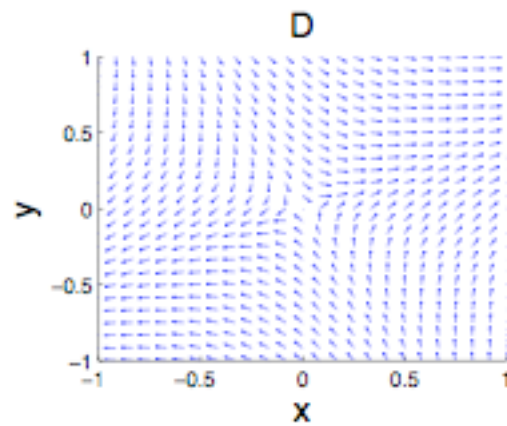
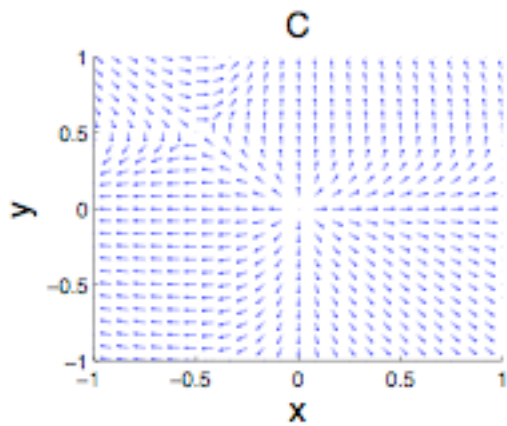
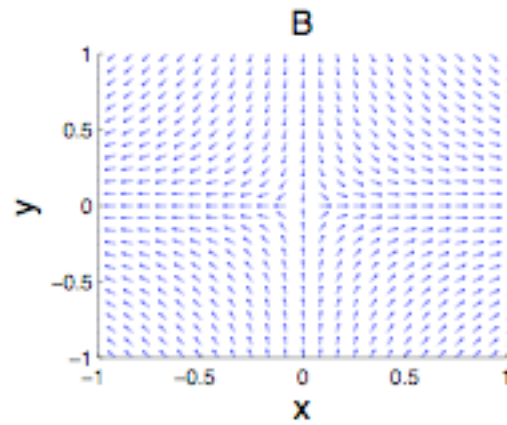
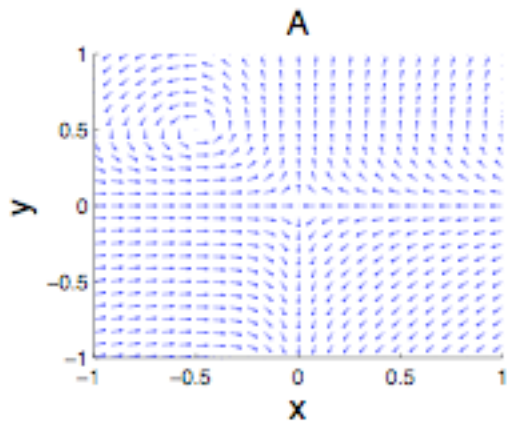
(c) For $\lambda_1 = \frac{-1}{2} + \frac{\sqrt{29}}{2}$ the eigenvector (up to scalar multiplication) is

$$v_1 = \begin{bmatrix} \frac{2}{\sqrt{29} - 5} \\ 1 \end{bmatrix}$$

$$\text{For } \lambda_2 = \frac{-1}{2} - \frac{\sqrt{29}}{2} \text{ the eigenvector is } v_2 = \begin{bmatrix} \frac{2}{\sqrt{29} + 5} \\ 1 \end{bmatrix}.$$

- (d) In general, an equilibrium solution is obtained at the intersection of nullclines of all the equations. In this case, the equilibrium point is at the origin. We can determine the stability structure of the origin by the eigenvalues. Since one of them is negative and the other positive the origin is a saddle point.

6. Match the following equations (1)-(4) with their corresponding direction fields A-D. You do not need to show your work for this problem. Write your answers of your exam sheet.



ANSWER:

- (1) - D
- (2) - B
- (3) - C
- (4) - A

7. (a) $y(t) = \cos t$ is an equilibrium solution of $y'' + y = 0$ ☐ TRUE ☐ FALSE
- (b) Picard's Theorem guarantees local existence and uniqueness of a solution to the IVP $y' = y^{3/2}, y(0) = 0$. ☐ TRUE ☐ FALSE
- (c) The IVP $y' = y^{3/2}, y(0) = 0$ has a solution that is defined for all time. ☐ TRUE ☐ FALSE
- (d) If A and B are commuting matrices then $(AB)^2 = BA^2B$ ☐ TRUE ☐ FALSE
- (e) If A and B are $n \times n$ matrices and $\exists (AB)^{-1}$ then $A(AB)^{-1}B = I$. ☐ TRUE ☐ FALSE
- (f) The vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}^T, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}^T, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}^T \right\}$ span $\mathbb{R}^3$. ☐ TRUE ☐ FALSE
- (g) For three vectors in $\mathbb{R}^3$ to span $\mathbb{R}^3$ they must form a basis for $\mathbb{R}^3$ ☐ TRUE ☐ FALSE
- (h) The functions $(1+t)$ and $t(1+t)$ are linearly dependent. ☐ TRUE ☐ FALSE
- (i) $y_p = Ae^t$ is a suitable guess for the particular solution of $y'' - 2y' + 2y = e^t$. ☐ TRUE ☐ FALSE
- (j) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ has three linearly independent eigenvectors. ☐ TRUE ☐ FALSE
- (k) If A is a 3×3 real matrix whose eigenvalues $\{-1, i, -i\}$ then $\mathbf{x} = 0$ is a stable solution of the system of DEs $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ ☐ TRUE ☐ FALSE
- (l) One of the eigenvalues of the DE $y''' - y'' + 2y' - 2y = 0$ is $\sqrt{2}i$. ☐ TRUE ☐ FALSE
- (m) The system of DEs $x' = x + y, y' = x^4 - y^4$ has a line of equilibrium solutions in the (x, y) plane. ☐ TRUE ☐ FALSE
- (n) Let $x' = x + y, y' = x^4 - y^4$. The Jacobian matrix of the linearized system around $(x, y) = (1, -1)$ has eigenvalues $\{0, -5\}$ ☐ TRUE ☐ FALSE

SOLUTION:

- (a) Equilibrium solutions are constant solutions. Since $y(t) = \cos t$ is not constant in t the statement is FALSE.
- (b) $f(t, y) = y^{3/2}$ is defined at $(0, 0)$ so Picard's theorem guarantees existence around a neighborhood of the origin. $\frac{\partial}{\partial y}(y^{3/2}) = \frac{3}{2}\sqrt{y}$ is defined at the origin and for positive values of y , so Picard's theorem guarantees uniqueness as well. Hence, the statement is TRUE.
- (c) $y(t) = 0$ is a solution and defined for all time, so the statement is TRUE.
- Note that you would find this solution using Separation of Variables.
- (d) $(AB)^2 = ABAB$. Replacing the 1st AB with BA gives $BAAB = BA^2B$. TRUE.
- (e) $A(AB)^{-1}B = AB^{-1}A^{-1}B$. In general, this cannot be simplified to the identity matrix. Hence, the statement is FALSE.
- (f) The determinant of the three vectors is non-zero so they are linearly independent. Three linearly independent vectors in $\mathbb{R}^3$ must span it. Hence, the statement is TRUE.
- (g) When three vectors in $\mathbb{R}^3$ span it they must be linearly independent. Since they also span $\mathbb{R}^3$ they form a basis for it. Hence, the statement is TRUE.
- (h) The Wronskian is nonzero so they are linearly independent. Therefore, the statement is FALSE.
- (i) The roots of the homogeneous equation are complex, while e^t corresponds to $r = 1$. Hence, this is not an exceptional case. Since Ae^t is the standard guess the statement is TRUE.
- (j) A is upper triangular and its eigenvalues are on the main diagonal, that is, 1, 1 and 2. The repeated eigenvalue 1 has only one eigenvector (a deficient case). Therefore, the matrix has altogether only two linearly independent eigenvectors, not three. Hence, the statement is FALSE.
- (k) For stability the roots should be negative or complex conjugates with a negative real part. In this case there is a negative root -1 and complex conjugates $-i$ and i , which are neutral and do not affect the stability. Hence, the statement is TRUE.

- (l) Substituting $y = e^{rt}$ we get $r^3 - r^2 + 2r - 2 = 0$. You can check that $r = \sqrt{2}i$ does not satisfy the equation. Hence, the statement is FALSE.
- (m) Equilibrium solutions are obtained at the intersection of two different nullclines. Here the x and y nullclines are the solutions to

$$x' = x + y = 0$$

$$y' = x^4 - y^4 = (x^2 - y^2)(x^2 + y^2) = (x + y)(x - y)(x^2 + y^2) = 0$$

We see that the x nullcline is the line $y = -x$ and that the y nullclines are the lines $y = x$, $y = -x$ (the third factor is strictly positive except at the origin). From this we can conclude that $y = -x$ is a line of equilibrium points. Hence, the statement is TRUE.

(n) The Jacobian is $J(x, y) = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 1 \\ 4x^3 & -4y^3 \end{vmatrix}$.

At $(1, -1)$ we get the characteristic equation as follows:

$$J(1, -1) = \begin{vmatrix} 1 & 1 \\ 4 & 4 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 - \lambda & 1 \\ 4 & 4 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 5\lambda = 0 \Rightarrow \lambda(\lambda - 5) = 0.$$

So there are two eigenvalues: $\lambda_1 = 0$ & $\lambda_2 = 5$, which differ from the eigenvalues stated in the statement, 0 and -5. Hence, the statement is FALSE.