**Duration: 3 hours**

Instructions: Answer all questions, without the use of notes or books. Calculators may be used to calculate numbers only. Partial credit will be awarded for correct work, unless otherwise specified. The total number of points is 100.

1. (7 points) Suppose that $X_1, \ldots, X_n$ are independent exponential random variables with parameter $\lambda$. That is, they all have the same probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Determine the maximum likelihood estimator of $\lambda$. Show your argument.

2. (8 points total) You asked your neighbor to water a sickly plant while you are on vacation. Without water it will die with probability .8; with water it will die with probability .15. You are 90% certain that your neighbor will remember to water the plant.

   (a) (5 points) What is the probability that the plant will be alive when you return?

   (b) (3 points) If it is dead, what is the probability your neighbor forgot to water it?

3. (8 points) A sample of 20 cigarettes is tested to determine nicotine content and the average value observed was 1.2 mg.

   (a) (4 points) Compute a 99 percent two–sided confidence interval for the mean nicotine content of a cigarette if it is known that the standard deviation of a cigarette's nicotine content is $\sigma = .2$ mg.

   (b) (4 points) Suppose that the population variance is not known in advance of the experiment. If the sample variance is .04, compute a value $c$ fro which we can assert "with 99 percent confidence" that $c$ is lager than the mean nicotine content of a cigarette.

4. (7 points) The lifetime of a certain electrical part is a random variable with mean 100 hours and standard deviation 20 hours. If 16 such parts are tested, use the Central Limit Theorem to approximate the probability that the sample mean is less than 104.

5. (12 points: 3 each) A college professor never finishes his lecture before the end of the hour and always finishes his lectures within 2 min after the hour. Let $X =$ the time, measured in minutes, that elapses between the end of the hour and the end of the lecture and suppose the probability density function of $X$ is

$$f(x) = \begin{cases} kx^2 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

   (a) Find the value of $k$.

   (b) What is the probability that the lecture continues beyond the hour for between 1 and 1.5 min?

   (c) Calculate $E[X]$.

   (d) Calculate $\mathrm{Var}(X)$.

6. (10 points total) Suppose the number $X$ of tornadoes observed in a particular region during a 1–year period has a Poisson distribution with $\lambda = 8$.

   (a) (5 points) What is the probability that there are less than 2 tornadoes observed in a given year?

   (b) (5 points) What is the probability that in 3 of the next 4 years there will be less than 2 tornadoes observed each year?

7. (8 points total)

   (a) (5 points) Jones figures that the total number of thousands of miles that a used auto can be driven before it would need to be junked is an exponential random variable with parameter $1/20$. Smith has a used car that he claims has been driven only 10,000 miles. If Jones purchases the car, what is the probability that she would get at least 20,000 additional miles out of it?

   (b) (3 points) Repeat under the assumption that the lifetime mileage of the car is not exponentially distributed but rather is (in thousands of miles) uniformly distributed over $(0, 40)$.

8. (7 points) $X$ is a Bernoulli random variable with parameter $p$. That is, $X$ may take on two values only, 0 and 1, and $P\{X = 1\} = p$ and $P\{X = 0\} = 1 - p$. Find the moment generating function of $X$ and use it to find $E[X]$ and $\mathrm{Var}(X)$.

9. (8 points total) The route used by a certain motorist in commuting to work has two intersections with traffic signals. The probability that he must stop at the first signal is 0.4. The probability that he must stop at the second signal is 0.5. The probability that he must stop at at least one of the signal is 0.6.

   (a) (4 points) What is the probability that he must stop at both signals?

   (b) (4 points) Are "stopping at the first signal" and "stopping at the second signal" independent events and why?

10. (15 points: 3 each) Only your final $\boxed{\text{boxed}}$ answers will be graded for the following problems.

    (a) From past experience, a professor knows that the test score of a student taking her final examination is a random variable with mean 75 and standard deviation 5. What can be said about the probability that a student will score between 65 and 85? (Hint: Chebyshev's inequality)

    (b) A data set consists of three pairs of values: $(3, 6)$, $(5, 7)$, and $(1, 5)$. Without calculation, do you think that the correlation coefficient is positive or negative, its absolute value is closer to 0 or closer to 1? Explain your reasoning in proper English.

    (c) Let $F$ be the event that it is Friday and $A$ be the event that John is absent from school. Explain in English the meaning of the notation $P(F|A^c)$.

    (d) Are $X$ and $Y$ independent if their joint probability density function is given by

    $$f(x, y) = \begin{cases} \dfrac{1 + xy}{4}, & |x| < 1 \text{ and } |y| < 1 \\ 0, & \text{otherwise} \end{cases}$$

    (e) If $E[X] = 2$ and $E[X^2] = 8$, calculate $E[(2 + 4X)^2]$.

11. ( points: 2 each) Determine each of the following statements True or False. No reasoning is needed. Please spell out the entire word "True" or "False".

    (a) $P(EF^c) = P(E) - P(EF)$ for any events $E$ and $F$.

    (b) $P(EF) = P(E)P(F)$ for any events $E$ and $F$.

    (c) If $X$ is a Poisson random variable with parameter $\lambda = 3$, then $2X = X + X$ is also a Poisson random variable with parameter $3 + 3 = 6$.

    (d) The sample mean $\overline{X}$ is always a normal random variable according to the central limit theorem.

    (e) If $\mathrm{Var}(X) = 3$ and $\mathrm{Var}(Y) = 4$, than $\mathrm{Var}(X + Y) = 7$.

**TABLE A3** *Values of* $t_{\alpha,n}$

| $n$ | $\alpha = .10$ | $\alpha = .05$ | $\alpha = .025$ | $\alpha = .01$ | $\alpha = .005$ |
|---|---|---|---|---|---|
| 1 | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 |
| 2 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 |
| 3 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 |
| 4 | 1.533 | 2.132 | 2.776 | 3.474 | 4.604 |
| 5 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 |
| 6 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 |
| 7 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 |
| 8 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 |
| 9 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 |
| 10 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 |
| 11 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 |
| 12 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 |
| 13 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 |
| 14 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 |
| 15 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 |
| 16 | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 |
| 17 | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 |
| 18 | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 |
| 19 | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 |
| 20 | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 |
| 21 | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 |
| 22 | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 |
| 23 | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 |
| 24 | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 |
| 25 | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 |
| 26 | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 |
| 27 | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 |
| 28 | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 |
| 29 | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 |
| $\infty$ | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 |

*Other t Probabilities:*
$P\{T_8 < 2.541\} = .9825$ $P\{T_8 < 2.7\} = .9864$ $P\{T_{11} < .7635\} = .77$ $P\{T_{11} < .934\} = .81$ $P\{T_{11} < 1.66\} = .94$ $P\{T_{12} < 2.8\} = .984$.

**TABLE A1** *Standard Normal Distribution Function:* $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$

| $x$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| .1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| .2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| .3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| .4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| .5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| .6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| .7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| .8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| .9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |