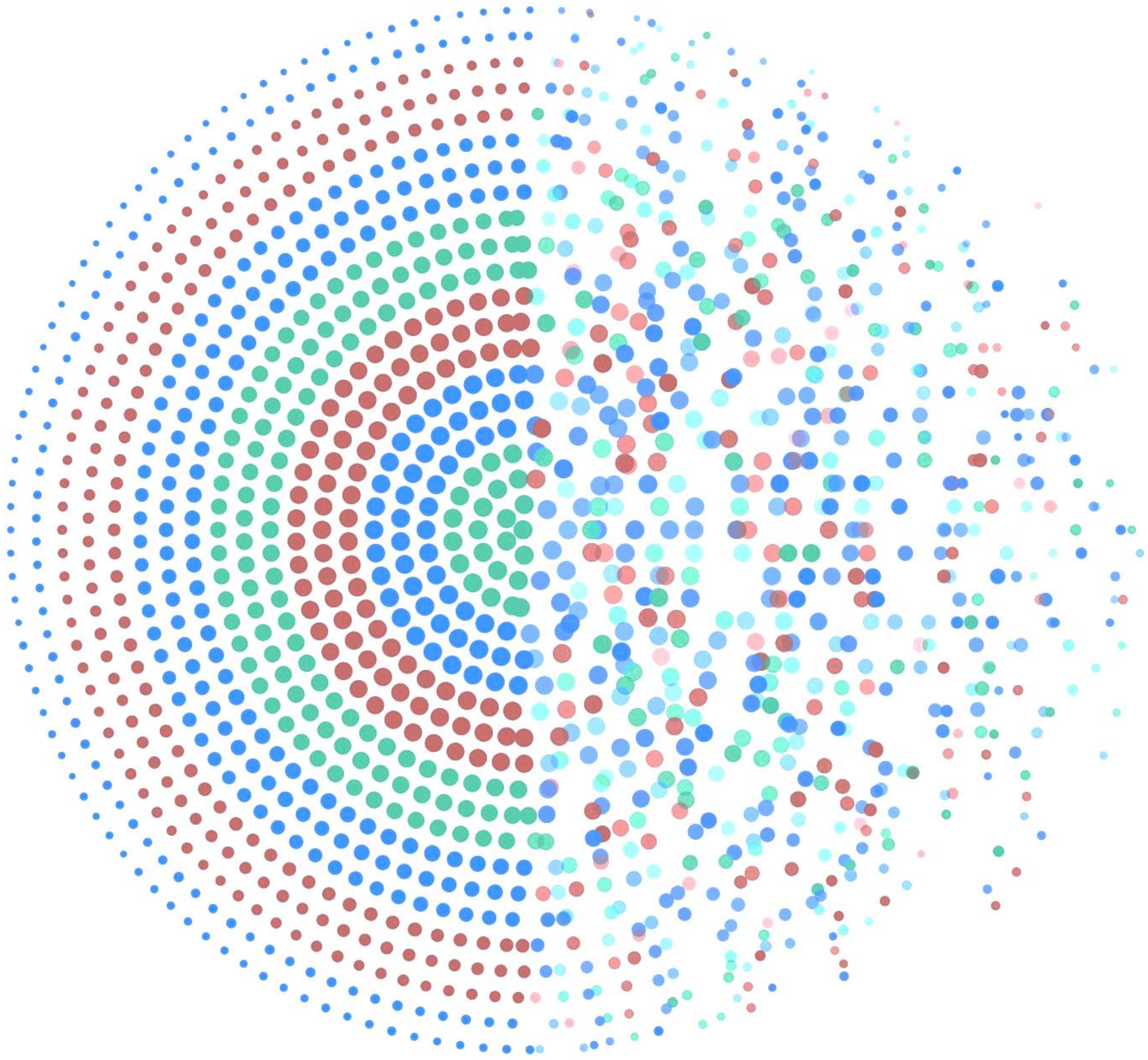
MODELING SCATTERING PROBLEMS FOR MULTILAYERED MEDIA

E. A. CORTES

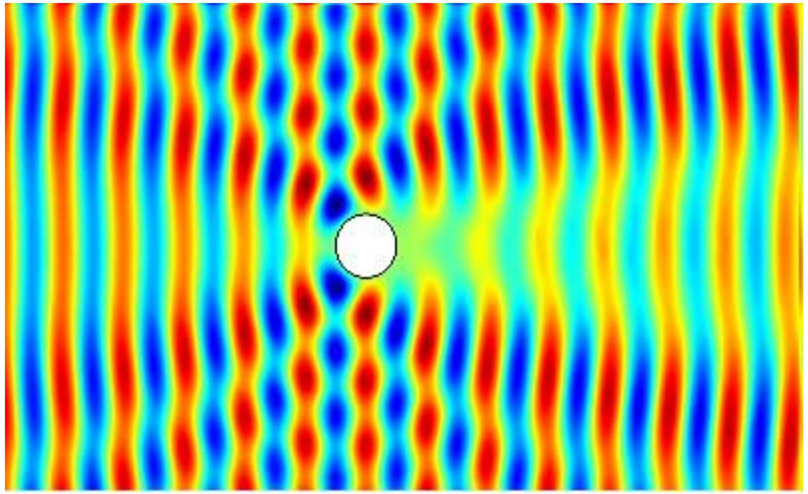
C. CARVALHO

C. TSOGKA



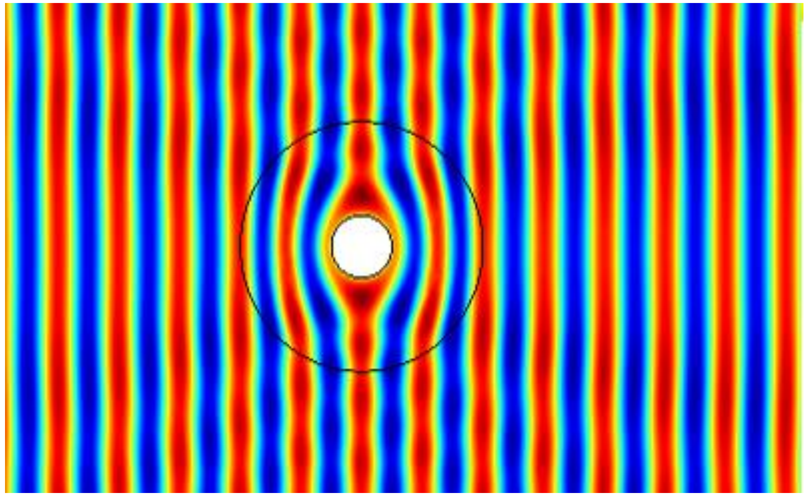
RESEARCH OBJECTIVES

- Develop a computational model for optical cloaking
  - Use boundary integral equations (BIE) to create this model
- Investigate parameters for effective cloaking - *optical properties*



Add coated layer(s)

Prevents scattering of source



Images Source: Physicsch on Wikipedia

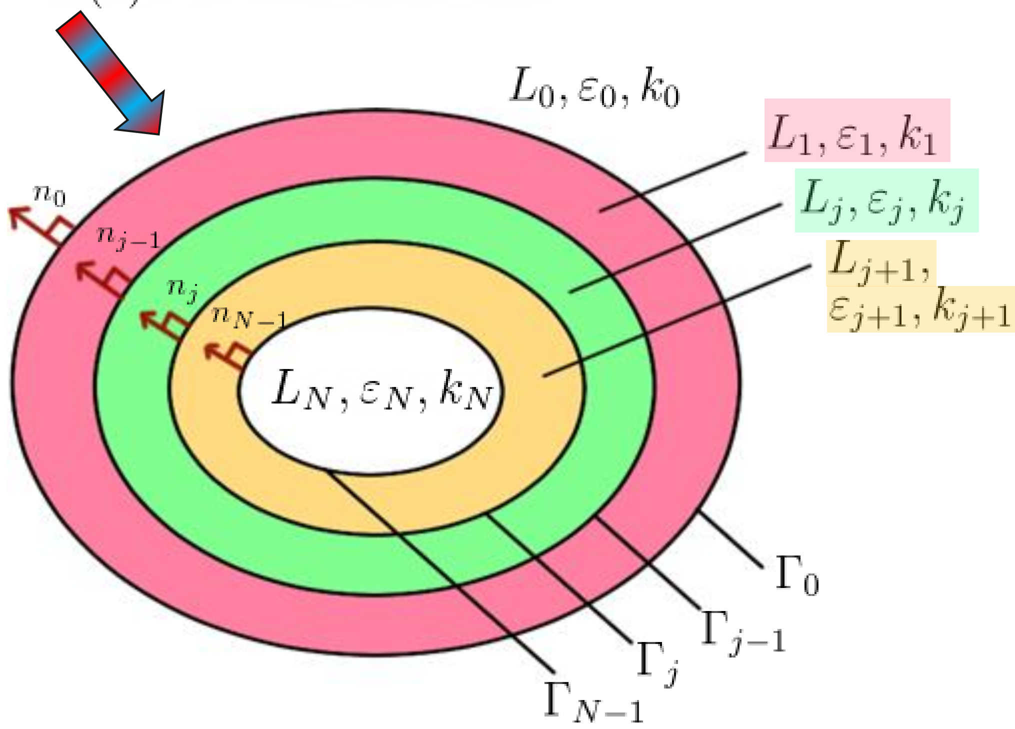
PROBLEM SETTING

Scattering Problem

$$div(\varepsilon^{-1}\nabla u) + \omega^2 u = 0$$

$$[u]_{\Gamma_j} = 0 \qquad [\varepsilon^{-1}\partial_n u]_{\Gamma_j} = 0 \qquad j = 0, \ldots, N-1$$

$$u = u^{in} + u^{sc}$$



$$k_j = \omega\sqrt{\varepsilon_j}$$

$$u_{i,j} = u_{\Gamma_i, L_j}$$

Transmission Problem

$\nabla u_{i,j} + k_j^2 u_{i,j} = 0$	in $\dot{L}_j = L_j \backslash \{\Gamma_j \cup \Gamma_{j-1}\}$	$j = 0, \ldots, N$
$u_{j,j+1} - u_{j,j} = 0$	on Γ_j	$j = 0, \ldots, N-1$
$\frac{1}{\varepsilon_{j+1}}\partial_{n_j} u_{j,j+1} - \frac{1}{\varepsilon_j}\partial_{n_j} u_{j,j} = 0$	on Γ_j	$j = 0, \ldots, N-1$

FORMULATION OF BIE SYSTEM

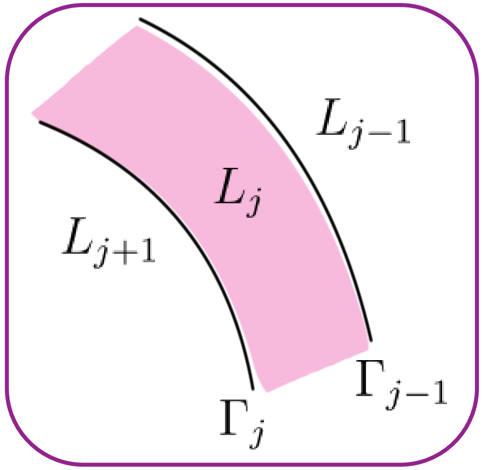
We use the fundamental solution to the Helmholtz equation

$$\Phi_j(x,y) = \frac{i}{4} H_0^{(1)}(k_j|x-y|)$$

This gives us the following layer potentials

$$D_{i,j}[\mu](x) = \int_{\Gamma_i} \frac{\partial \Phi_j}{\partial n_i}(x,y)\mu(y)d\sigma_y \qquad x \in \dot{L}_j \qquad \Gamma_i = \Gamma_{j-1}, \Gamma_j$$

$$S_{i,j}[\mu](x) = \int_{\Gamma_i} \Phi_j(x,y)\mu(y)d\sigma_y \qquad x \in \dot{L}_j \qquad \Gamma_i = \Gamma_{j-1}, \Gamma_j$$



We use these layer potentials to represent the solution in each layer.

The density is based on the data of the boundary we are integrating over, so $y = x_i^b, \quad x_i^b = x \in \Gamma_i$

2 LAYER SYSTEM

Representation Formulas

$$u_{0,0}(x) = u^{in}(x) + D_{0,0}[u_{0,0}](x) - S_{0,0}[\partial_{n_0} u_{0,0}](x) \qquad x \in \dot{L}_0$$

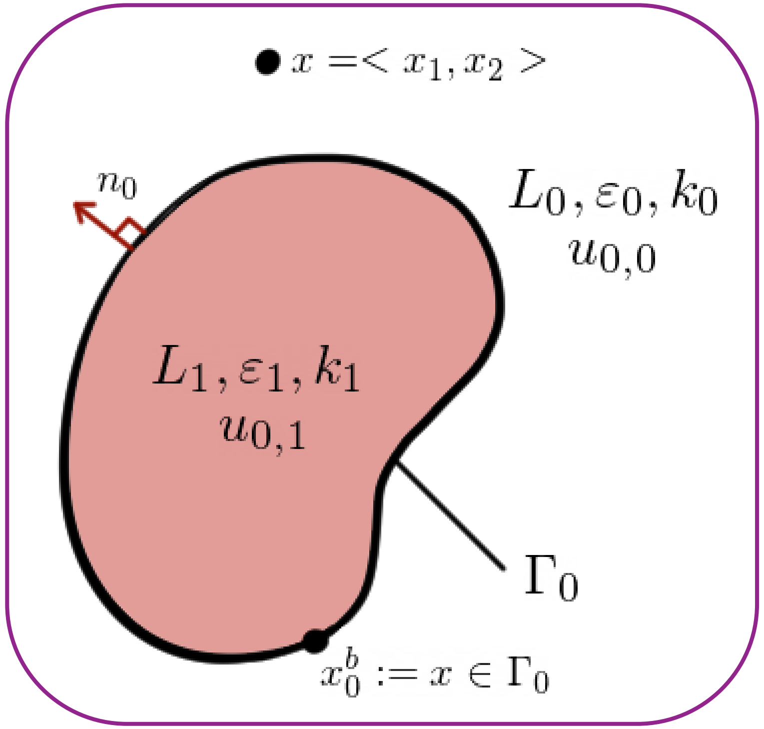
$$u_{0,1}(x) = S_{0,1}[\partial_{n_0} u_{0,1}](x) - D_{0,1}[u_{0,1}](x) \qquad x \in \dot{L}_1$$

$$D[\mu](x) \underset{x \to x_b^\pm}{\longrightarrow} \pm\frac{1}{2}\mu(x^b) + D[\mu](x^b)$$

$$S[\mu](x) \underset{x \to x_b^\pm}{\longrightarrow} S[\mu](x^b)$$

$$\frac{1}{2}u_{0,0}(x_0^b) = u^{in}(x_0^b) + D_{0,0}[u_{0,0}](x_0^b) - S_{0,0}[\partial_{n_0} u_{0,0}](x_0^b)$$

$$\frac{1}{2}u_{0,1}(x_0^b) = S_{0,1}[\partial_{n_0} u_{0,1}](x_0^b) - D_{0,1}[u_{0,1}](x_0^b)$$



$$\frac{1}{2}u_{0,0}(x_0^b) = u^{in}(x_0^b) + D_{0,0}[u_{0,0}](x_0^b) - S_{0,0}[\partial_{n_0}u_{0,0}](x_0^b)$$

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$$\frac{1}{2}u_{0,0}(x_0^b) = \frac{\varepsilon_1}{\varepsilon_0}S_{0,1}[\partial_{n_0}u_{0,0}](x_0^b) - D_{0,1}[u_{0,0}](x_0^b)$$

Transmission Conditions

$$u_{0,1} - u_{0,0} = 0 \qquad x \in \Gamma_0$$

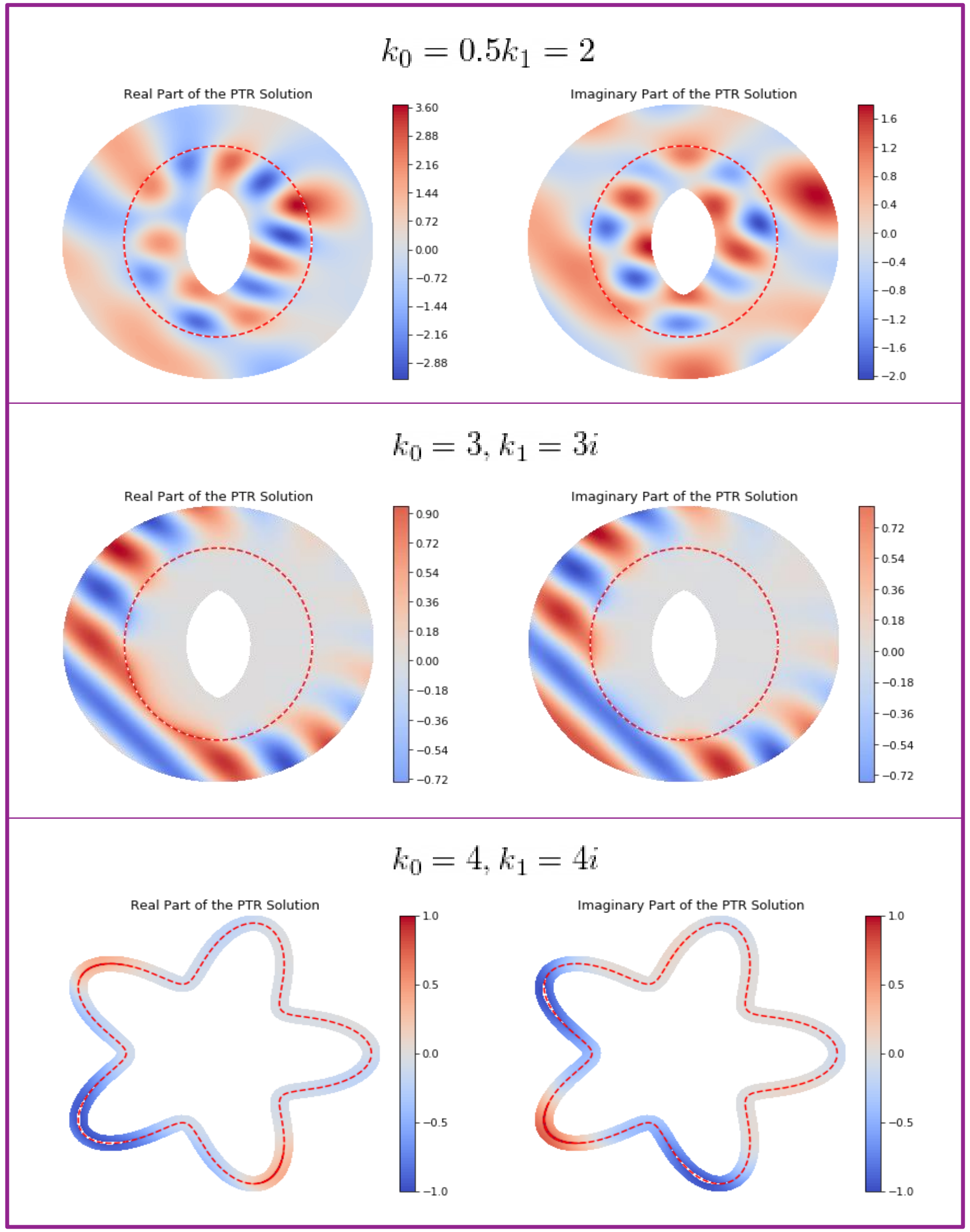
$$\frac{1}{\varepsilon_1}\partial n_0 u_{0,1} - \frac{1}{\varepsilon_0}\partial_{n_0}u_{0,0} = 0 \qquad x \in \Gamma_0$$

$$\begin{bmatrix} \frac{I}{2} - D_{0,0} & S_{0,0} \\ \frac{I}{2} + D_{0,1} & \frac{\varepsilon_1}{\varepsilon_0}S_{0,1} \end{bmatrix} \begin{bmatrix} u_{0,0} \\ \partial_{n_0}u_{0,0} \end{bmatrix} = \begin{bmatrix} u^{in}(x_0^b) \\ 0 \end{bmatrix}$$

We end up with a solvable BIE System

RESULTS FOR TRANSMISSION

- Step 1 – Evaluate layer potentials for the BIE system
  - Evaluated using the *Kress Quadrature*
  - Treats singularity that occurs in Hankel function for $|x - y| = 0$
- Step 2 – Find boundary data using the BIE system
- Step 3 – Use boundary data to represent the solution in each layer
  - Solution layers are computed using the *Periodic Trapezoid Rule (PTR)* approximation
- We can produce results for various layer properties and boundary shapes



3 LAYER SYSTEM

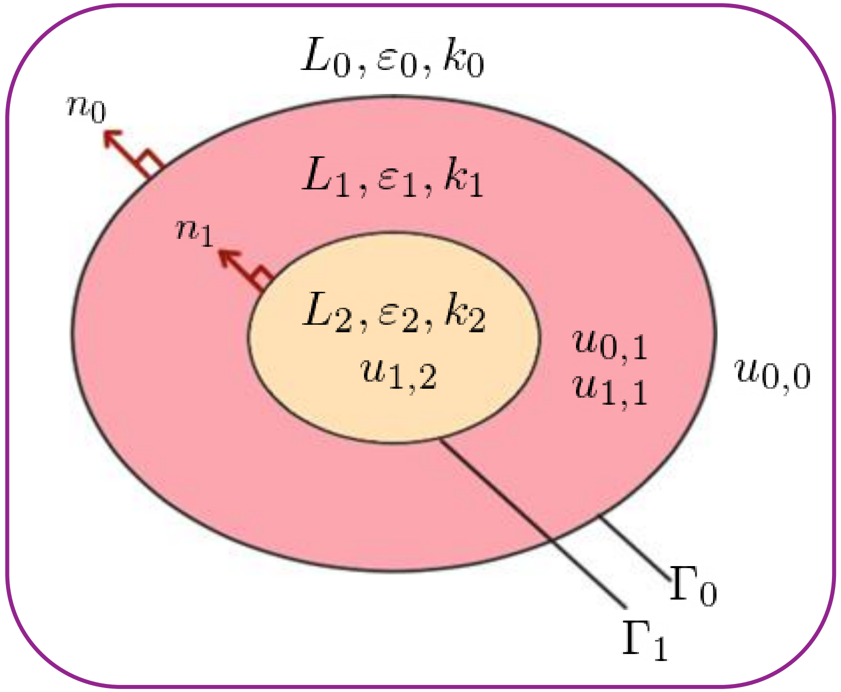
Representation Formulas

$$u_{0,0}(x) = u^{in}(x) + D_{0,0}[u_{0,0}](x) - S_{0,0}[\partial_{n_0} u_{0,0}](x) \qquad x \in \dot{L}_0$$

$$u_{0,1}(x) = S_{0,1}[\partial_{n_0} u_{0,1}](x) - D_{0,1}[u_{0,1}](x) \qquad x \in \dot{L}_1$$

$$u_{1,1}(x) = u_1^{\infty}(x) + D_{1,1}[u_{1,1}](x) - S_{1,1}[\partial_{n_1} u_{1,1}](x) \qquad x \in \dot{L}_1$$

$$u_{1,2}(x) = S_{1,2}[\partial_{n_1} u_{1,2}](x) - D_{1,2}[u_{1,2}](x) \qquad x \in \dot{L}_2$$



$$\frac{1}{2} u_{0,0}(x_0^b) = u^{in}(x_0^b) + D_{0,0}[u_{0,0}](x_0^b) - S_{0,0}[\partial_{n_0} u_{0,0}](x_0^b)$$

$$\frac{1}{2} u_{0,1}(x_0^b) = S_{0,1}[\partial_{n_0} u_{0,1}](x_0^b) - D_{0,1}[u_{0,1}](x_0^b)$$

$$\frac{1}{2} u_{1,1}(x_1^b) = u_1^{\infty}(x_1^b) + D_{1,1}[u_{1,1}](x_1^b) - S_{1,1}[\partial_{n_1} u_{1,1}](x_1^b) \qquad u_1^{\infty}(x_1^b) = u_{0,0}(x_1^b)$$

$$\frac{1}{2} u_{1,2}(x_1^b) = S_{1,2}[\partial_{n_1} u_{1,2}](x_1^b) - D_{1,2}[u_{1,2}](x_1^b)$$

$$u_1^\infty(x_1^b) = u_{0,0}(x_1^b) = u^{in}(x_1^b) + D_{0,0}[u_{0,0}](x_1^b) - S_{0,0}[\partial_{n_0} u_{0,0}](x_1^b)$$

$$\left.\begin{aligned} \frac{1}{2}u_{0,0}(x_0^b) &= u^{in}(x_0^b) + D_{0,0}[u_{0,0}](x_0^b) - S_{0,0}[\partial_{n_0} u_{0,0}](x_0^b) \\ \frac{1}{2}u_{0,1}(x_0^b) &= S_{0,1}[\partial_{n_0} u_{0,1}](x_0^b) - D_{0,1}[u_{0,1}](x_0^b) \end{aligned}\right\} x_0^b$$

$$\left.\begin{aligned} \frac{1}{2}u_{1,1}(x_1^b) &= u^{in}(x_1^b) + D_{0,0}[u_{0,0}](x_1^b) - S_{0,0}[\partial_{n_0} u_{0,0}](x_1^b) + D_{1,1}[u_{1,1}](x_1^b) - S_{1,1}[\partial_{n_1} u_{1,1}](x_1^b) \\ \frac{1}{2}u_{1,2}(x_1^b) &= S_{1,2}[\partial_{n_1} u_{1,2}](x_1^b) - D_{1,2}[u_{1,2}](x_1^b) \end{aligned}\right\} x_1^b$$

$$u_{0,1} - u_{0,0} = 0 \qquad x \in \Gamma_0$$

$$\frac{1}{\varepsilon_1}\partial n_0 u_{0,1} - \frac{1}{\varepsilon_0}\partial_{n_0} u_{0,0} = 0 \qquad x \in \Gamma_0$$

$$u_{1,2} - u_{1,1} = 0 \qquad x \in \Gamma_1$$

$$\frac{1}{\varepsilon_2}\partial n_1 u_{1,2} - \frac{1}{\varepsilon_1}\partial_{n_1} u_{1,1} = 0 \qquad x \in \Gamma_1$$

Transmission Conditions

$$\frac{1}{2}u_{0,0}(x_0^b) = u^{in}(x_0^b) + D_{0,0}[u_{0,0}](x_0^b) - S_{0,0}[\partial_{n_0}u_{0,0}](x_0^b)$$

$$\frac{1}{2}u_{0,0}(x_0^b) = \frac{\varepsilon_1}{\varepsilon_0}S_{0,1}[\partial_{n_0}u_{0,0}](x_0^b) - D_{0,1}[u_{0,0}](x_0^b)$$

$$\frac{1}{2}u_{1,1}(x_1^b) = u^{in}(x_1^b) + D_{0,0}[u_{0,0}](x_1^b) - S_{0,0}[\partial_{n_0}u_{0,0}](x_1^b) + D_{1,1}[u_{1,1}](x_1^b) - S_{1,1}[\partial_{n_1}u_{1,1}](x_1^b)$$

$$\frac{1}{2}u_{1,1}(x_1^b) = \frac{\varepsilon_2}{\varepsilon_1}S_{1,2}[\partial_{n_1}u_{1,1}](x_1^b) - D_{1,2}[u_{1,1}](x_1^b)$$

$$\begin{matrix} x_0^b \\ \\ x_1^b \end{matrix} \begin{bmatrix} \frac{I}{2} - D_{0,0} & S_{0,0} & 0 & 0 \\ \frac{I}{2} + D_{0,1} & \frac{\varepsilon_1}{\varepsilon_0}S_{0,1} & 0 & 0 \\ -D_{0,0} & S_{0,0} & \frac{I}{2} - D_{1,1} & S_{1,1} \\ 0 & 0 & \frac{I}{2} + D_{1,2} & \frac{\varepsilon_2}{\varepsilon_1}S_{1,2} \end{bmatrix} \begin{bmatrix} u_{0,0} \\ \partial_{n_0}u_{0,0} \\ u_{1,1} \\ \partial_{n_1}u_{1,1} \end{bmatrix} = \begin{bmatrix} u^{in}(x_0^b) \\ 0 \\ u^{in}(x_1^b) \\ 0 \end{bmatrix} \begin{matrix} x_0^b \\ \\ x_1^b \end{matrix}$$

Solvable BIE System of Coupled Equations

MULTILAYER SYSTEM

Representation Formulas

$$u_{0,0}(x) = u^{in}(x) + D_{0,0}[u_{0,0}](x) - S_{0,0}[\partial_{n_0} u_{0,0}](x) \qquad x \in \dot{L}_0$$

$$u_{j-1,j}(x) = S_{j-1,j}[\partial_{n_{j-1}} u_{j-1,j}](x) - D_{j-1,j}[u_{j-1,j}](x) \qquad x \in \dot{L}_j \qquad j = 1, \ldots, N$$

$$u_{j,j}(x) = u_j^{\infty}(x) + D_{j,j}[u_{j,j}](x) - S_{j,j}[\partial_{n_j} u_{j,j}](x) \qquad x \in \dot{L}_j \qquad j = 1, \ldots, N-1$$

⬇

$$\frac{1}{2} u_{0,0}(x_0^b) = u^{in}(x_0^b) + D_{0,0}[u_{0,0}](x_0^b) - S_{0,0}[\partial_{n_0} u_{0,0}](x_0^b)$$

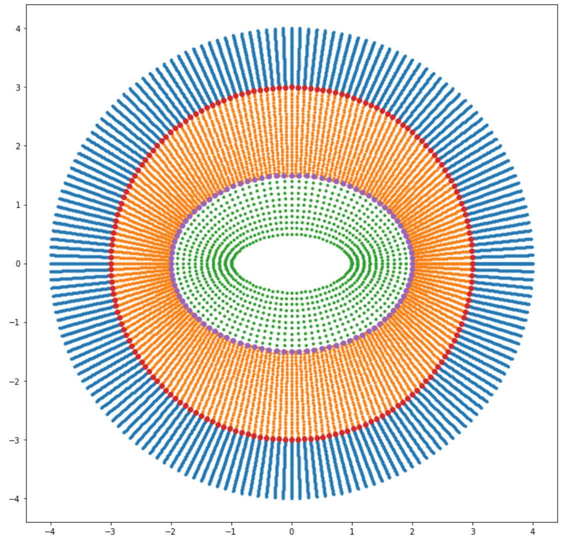
$$\frac{1}{2} u_{j-1,j}(x_{j-1}^b) = S_{j-1,j}[\partial_{n_{j-1}} u_{j-1,j}](x_{j-1}^b) - D_{j-1,j}[u_{j-1,j}](x_{j-1}^b) \qquad j = 1, \ldots, N$$

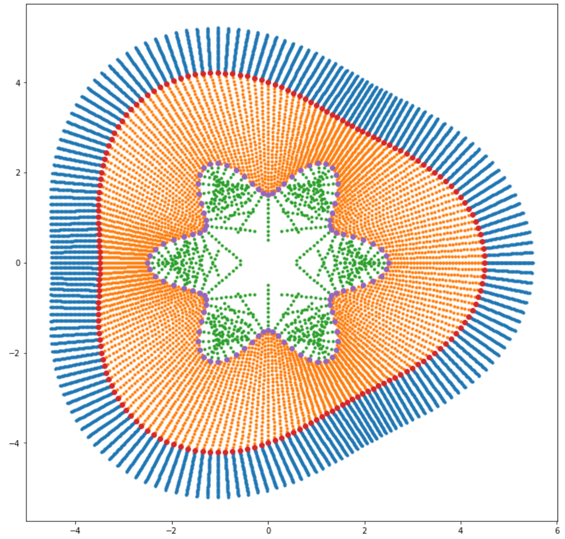
$$\frac{1}{2} u_{j,j}(x_j^b) = u_j^{\infty}(x_j^b) + D_{j,j}[u_{j,j}](x_j^b) - S_{j,j}[\partial_{n_j} u_{j,j}](x_j^b) \qquad j = 1, \ldots, N-1$$

$$u_j^{\infty}(x) = u_{j-1,j-1}(x)$$

FUTURE WORK

- Code to model these results is still in development
- If results are satisfactory, we want to extend this work further
  - Boundaries of different shapes layered together
  - Layers with different properties
  - Models in 3D, models that change over time
- Other Challenges
  - Close evaluation problem - need to treat as we deal with smaller layers





QUESTIONS?