Applied Mathematics Preliminary Exam: Linear Algebra

University of California, Merced
January 2026

Instructions

Read the following instructions carefully:

- Write your name on the front page of your exam.
- This is a closed book exam. You are allowed a single two-sided 8.5×11 inch sheet of paper on which you may write whatever you want. (Only hand written material allowed.) No phones or calculators are to be used during the exam.
- Write each problem on a separate page. Please make sure you clearly mark the problem you are working on (e.g., 1a).
- It is important to show your work for each problem. Credit will NOT be given for correct answers without justification. Also, partial credit will be given for incorrect answers if some of the work is correct.
- Clearly mark out (cross out) any work that you are not including in your answer and you do not want graded.
- Be sure to staple your exam at the end and hand it in.
- Good luck!

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Friday, January 16, 2026, 9:00am–1:00pm

1. (16 Points; 4 Points Each Part) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 6 & 2 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

   (a) Find a basis for the null space $N(A)$.

   (b) Find the general solution to $A\vec{x} = \vec{b}$, including determining when a solution exists.

   (c) Find a basis for the column space of A and explain how you found it.

   (d) Find the rank of A^T and explain how you found it.

2. (12 Points; 4 Points Each Part) Let t be a real number. Consider the matrix

$$A = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}.$$

   (a) Explain why you can tell, without calculating, that all the eigenvalues of A will be real. (Provide a short justification for your explanation.)

   (b) For what values of t will all the eigenvalues of A be positive?

   (c) For what values of t will A be invertible?

3. (16 Points; 4 Points Each Part) Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be nonzero vectors and define

$$A = \vec{u}\vec{v}^T.$$

   (a) What is the rank of A? Explain how you know this.

   (b) Determine all eigenvalues of A, provide a short justification for your conclusions.

   (c) For each eigenvalue of A, determine the corresponding eigenvectors.

   (d) Determine necessary and sufficient conditions on $\vec{u}$ and $\vec{v}$ such that

$$A^2 = 0.$$

4. (12 Points; 4 Points Each Part) Consider the set V of all 3×3 matrices A that are diagonalized by the same fixed invertible matrix S.

   (a) Prove that V is a vector space and report its dimension.

   (b) Write a basis for this vector space (your basis vectors will be in terms of S). Justify this is a basis by showing that the set of vectors you found is linearly independent and spans V.

   (c) Let W to be the subset of matrices in V with determinant 0. That is:

$$W = \{A \in V | \det(A) = 0\}.$$

   Is W a vector space? Why or why not?

5. (12 Points; 4 Points Each Part) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

(a) Perform the Gram-Schmidt process to create an orthonormal basis that has the same span as the columns of A

(b) Compute the QR factorization of the matrix A. (Your first part might be helpful in this.)

(c) Use any method you like to determine the least squares solution to

$$A\vec{x} = \vec{b}.$$

6. (8 Points; 4 Points Each Part) In this problem consider orthogonal matrices under different matrix norms: the matrix 2-norm, and the Frobenius norm.

- Matrix 2-norm:
$$\|A\|_2 = \max_{\vec{x} \neq \vec{0}} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2} = \max_{\|\vec{x}\|_2 = 1} \|A\vec{x}\|_2.$$

- Frobenius norm:
$$\|A\|_F = \sqrt{\sum_{i,j} (a_{ij})^2}$$
where $a_{i,j}$ is the value of matrix A at row i and column j. Notice that you can think of this as the Euclidean norm of the matrix when treated as a vector in $\mathbb{R}^{n^2}$.

Let $Q \in \mathbb{R}^{n \times n}$ be an orthogonal matrix and consider the matrix norms we defined above. Surprisingly, for each of these specific norms all $n \times n$ orthogonal matrices Q have the same norm.

(a) Prove that $\|Q\|_2 = 1$.

(b) Prove that $\|Q\|_F = \sqrt{n}$.

7. (16 Points; 4 Points Each Part) Answer each of the following questions briefly. Explain your reasoning in 1–2 complete sentences.

(a) Suppose $\vec{v}$ is an eigenvector of $A \in \mathbb{R}^{n \times n}$ with eigenvalue λ. Let $B = 3A - 2I_n$. Determine an eigenvector/eigenvalue pair of B. Explain your reasoning.

(b) Suppose $\vec{u}_1, \vec{u}_2, \vec{u}_3 \in \mathbb{R}^4$ are linearly independent with $\vec{u}_1 \perp \vec{u}_2$ and $\vec{u}_1 \perp \vec{u}_3$. True or false: $\vec{u}_2 \perp \vec{u}_3$? Explain.

(c) True or False: If $A\vec{x} = \vec{b}$ always has at least one solution, then the only solution to $A^T\vec{y} = \vec{0}$ is $\vec{y} = \vec{0}$. Explain.

(d) Suppose A is a 2×2 symmetric matrix with unit eigenvectors $\vec{u_1}$ and $\vec{u_2}$ corresponding to eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. Write its singular value decomposition: $A = U\Sigma V^T$.