Applied Mathematics Preliminary Exam: Ordinary Differential Equations

University of California, Merced
January 2026

Instructions

Read the following instructions carefully:

- Write your name on the front page of your exam.
- You may use a single, double-sided 8.5×11 inch sheet of your own hand-written notes for this exam. No other notes, books, or calculators are permitted.
- Write each problem on a separate page. Please make sure you clearly mark the problem you are working on (e.g., 1a).
- It is important to show your work for each problem. Credit will NOT be given for correct answers without justification. Also, partial credit will be given for incorrect answers if some of the work is correct.
- Clearly mark out (cross out) any work that you are not including in your answer and you do not want graded.
- Be sure to staple your exam at the end and hand it in.
- Good luck!

Applied Mathematics Preliminary Exam: Ordinary Differential Equations

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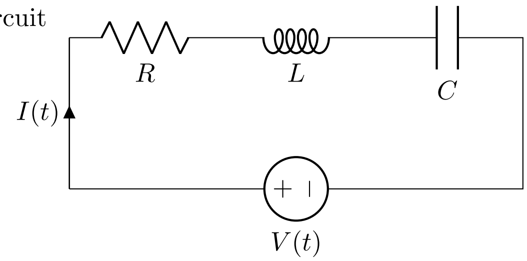
Wednesday, January 14, 2026, 9am-1pm

1. (20 Points) Consider the circuit



with **resistance R, inductance L, and capacitance C. All three (R, L, C) are positive.** $I(t)$ is the current flowing through the circuit. By Kirchoff's voltage law, the voltage across the resistor, inductor, and capacitor must equal the source voltage $V(t)$:

$$V_R(t) + V_L(t) + V_C(t) = V(t).$$

With constant capacitance C, the voltage across the capacitor is $V_C(t) = Q(t)/C$ with charge $Q(t)$ related to the current via

$$I(t) = \frac{dQ}{dt}.$$

Furthermore, we have $V_L(t) = L dI/dt$ and $V_R(t) = RI(t)$ (Ohm's Law). Putting it all together, we obtain the following equation of motion for charge $Q(t)$:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q(t) = V(t). \tag{1}$$

The task is to give the solution to (1) under various conditions:

(a) (5 points) Assume that $R = 0$ and $V(t) \equiv 0$. Show that the solutions must be oscillatory with period $2\pi\sqrt{LC}$.

(b) (10 points) Continue assuming $V(t) \equiv 0$ but now assume $R > 0$. What values of R correspond to underdamped, critically damped, and overdamped solutions? Explain the general algebraic form of the solutions that result.

(c) (5 points) When $R = 0$ and $V(t) = \sin(t/\sqrt{LC})$, for initial conditions $Q(0) = 0$ and $dQ/dt(0) = 1$, find the solution $Q(t)$ and show that it grows in amplitude without bound.

2. (15 Points) Consider the following ODE:

$$\psi''(x) + (2E - x^2)\psi(x) = 0. \tag{2}$$

Here E is a real constant. There are three parts to this problem:

(a) (3 points) Let us first consider the particular case when $E = 1/2$. We obtain

$$\phi''(x) + \phi(x) - x^2\phi(x) = 0. \tag{3}$$

Here we choose the letter ϕ instead of ψ deliberately, since they will not be equal to one another for all values of E. **Consider a solution of the form**

$$\phi(x) = e^{f(x)}. \tag{4}$$

Substituting this into (3), carefully derive the following ODE for f:

$$(f'(x))^2 + f''(x) + 1 - x^2 = 0. \tag{5}$$

(b) (2 points) **Try a polynomial solution of the form**

$$f(x) = Ax^2 + Bx.$$

What do you obtain for the constants A and B?

(c) (10 points) Now let us return to (2). Based on the results of (a) and (b), we try a solution of the form

$$\psi(x) = e^{-x^2/2}u(x).$$

Show that this results in the following ODE for $u(x)$:

$$\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + (2E-1)u = 0. \tag{6}$$

Try a power series solution of the form

$$u(x) = \sum_{n=0}^{\infty} \alpha_n x^n.$$

Show that if $E = N + 1/2$ for some nonnegative integer N, the power series terminates after finitely many terms, so that $u(x)$ ends up being a polynomial. In this case, explain why the overall solution $\psi(x)$ will satisfy this property:

$$\int_{x=-\infty}^{\infty} \psi(x)^2\,dx < \infty.$$

3. (35 Points) Consider the nonlinear system

$$\frac{dx}{dt} = y \tag{7a}$$

$$\frac{dy}{dt} = x(1+3x). \tag{7b}$$

This problem has three parts:

(a) (5 points) **Find all fixed points and determine their linear stability analytically, using the eigenvalues of the Jacobian.**

(b) (5 points) Define

$$E(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 - x^3.$$

Show that if $(x(t), y(t))$ solves (7), then

$$\frac{d}{dt}\left[E(x(t), y(t))\right] = 0.$$

Explain why this implies that

$$E(x(t), y(t)) = E(x(0), y(0)) \tag{8}$$

for all t. **What are the consequences of this fact for the phase portrait of the system?**

(c) (15 points) Consider the solution that satisfies the initial condition

$$(x(0), y(0)) = \left(-\frac{1}{2}, 0\right). \tag{9}$$

Using this initial condition in (8), derive

$$\frac{1}{2}y(t)^2 - \frac{1}{2}x(t)^2 - x(t)^3 = 0.$$

Since we assumed in the above derivation that $(x(t), y(t))$ solves (7), we must have $y(t) = dx/dt$, so the previous equation turns into a *first-order nonlinear ODE*:

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \frac{1}{2}x(t)^2 - x(t)^3 = 0.$$

Solve this ODE for $x(t)$ subject to the initial condition $x(0) = -1/2$. Hint: use the following antiderivative:

$$\int \frac{dx}{x\sqrt{1+2x}} = -2\tanh^{-1}\sqrt{1+2x} + C,$$

where tanh is hyperbolic tangent and $\tanh^{-1}$ is its inverse function. All you need to know about hyperbolic functions is either here or can be easily derived from what is here:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \; \cosh z = \frac{e^z + e^{-z}}{2}, \; \tanh z = \frac{\sinh z}{\cosh z}, \; 1 - \tanh^2 z = \operatorname{sech}^2 z = \frac{1}{\cosh^2 z}.$$

Once you have found $x(t)$, substitute it into (7a) to find the corresponding $y(t)$. You will then have found a particular solution $(x(t), y(t))$ of the *nonlinear system* (7)! Check that it satisfies the initial condition (9).

(d) (10 points) **Sketch a phase portrait of the nonlinear system (7). Your phase portrait should include fixed points and accurately represent their stability. Also explain why the particular solution $(x(t), y(t))$ you found in part (c) corresponds to a curve in the phase portrait that seemingly connects the fixed point at $(0, 0)$ back to itself!**

4. (30 Points) Let us begin with a coupled linear system in $\mathbb{R}^n$:

$$\frac{dx}{dt} = Ax(t).$$

Here A is an $n \times n$ matrix.

(a) (10 points) Suppose that the matrix A is diagonalizable, i.e., there exists an invertible matrix V (whose columns are the eigenvectors of A) and a diagonal matrix D such that

$$A = VDV^{-1}.$$

Using

$$y(t) = V^{-1}x(t),$$

show that $y(t)$ satisfies

$$\frac{dy}{dt} = Dy(t).$$

Given an arbitrary initial condition $y(0)$, solve this equation for $y(t)$.

(b) (10 points) **Continuing in the case where A is diagonalizable, use the above analysis to solve for $x(t)$ given an arbitrary initial condition $x(0)$. For full credit, your solution must be written in the form:**

$$x(t) = \Phi(t)x(0),$$

where $\Phi(t)$ is a matrix that equals the product of three matrices. The three matrices in question are allowed to depend on (or make use of the entries of) V, D, and time t.

(c) (10 points) Let us now add a forcing term $f(t)$ to the original equation:

$$\frac{dx}{dt} = Ax(t) + f(t). \tag{10}$$

Let us assume that $f(t)$ is a continuous curve taking values in $\mathbb{R}^n$.

Take the form of the solution $x(t)$ you derived in part (b) and replace the initial condition $x(0)$ with a time-dependent vector $z(t)$ so that you have

$$x(t) = \Phi(t)z(t).$$

Assume that $z(0) = x(0)$. **Substitute this new expression for $x(t)$ into (10) and carefully derive the solution for $z(t)$:**

$$z(t) = x(0) + \int_{s=0}^{t} \Phi(s)^{-1} f(s)\, ds.$$

From this, conclude that the solution of the forced equation (10) must be

$$x(t) = \Phi(t)x(0) + \Phi(t) \int_{s=0}^{t} \Phi(s)^{-1} f(s)\, ds.$$