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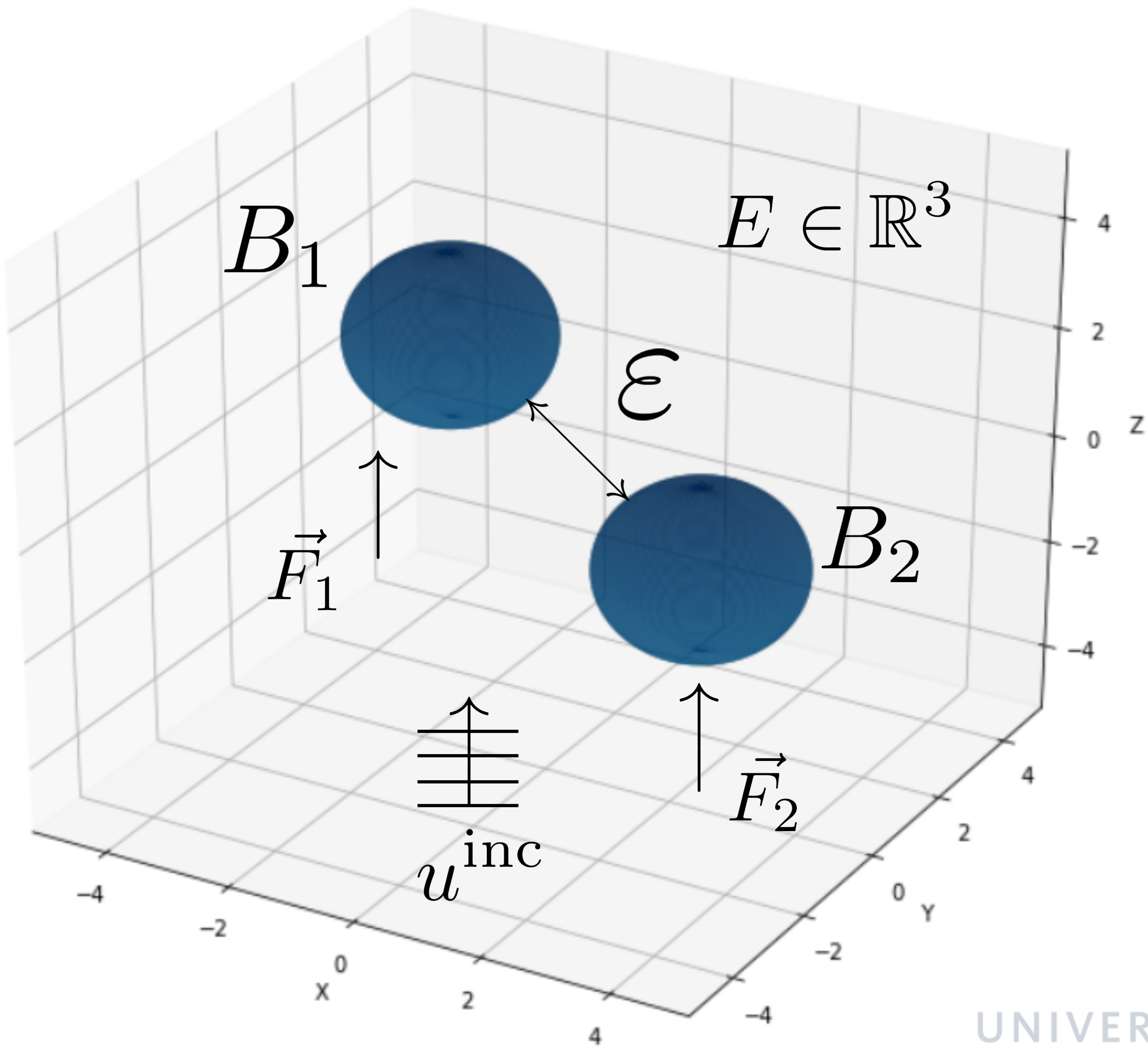
SCATTERING BY TWO CLOSELY SITUATED SOUND-HARD SPHERES.

PROBLEM DESCRIPTION



Project Goals:

- Find the solution of the scattering problem,

$$u := u^{\text{inc}} + u^{\text{sca}}.$$

$$\nabla^2 u + ku = 0, \qquad \text{in} \quad E$$

$$\partial_n u = \partial_n u^{\text{sca}} + \partial_n u^{\text{inc}} = 0, \qquad \text{on} \quad \partial E$$

$$\lim_{\xi \to \infty} \int_{|x| \to \xi} \left(\partial_n u^{\text{sca}} - iku^{\text{sca}}\right) ds = 0$$

$$u^{\text{inc}} = e^{ikz}$$

- Accurately compute solution when spheres are closely situated.

$$|\mathbf{x}_1 - \mathbf{x}_2| = 2a + \varepsilon, \quad \varepsilon \to 0^+$$

- Compute the acoustic radiation force.

$$\vec{F}_k = -\int_{B_k} \left\{ \left[\frac{1}{2}\kappa_0 \langle p_1^2 \rangle - \frac{1}{2}\rho_0 \langle v_1^2 \rangle \right] \hat{n} + \rho_0 \langle (\hat{n} \cdot \vec{v}_1)\vec{v}_1 \right\} dS, \quad k = 1, 2.$$

- Need to know the field, u on each sphere to calculate the force.

$$\vec{v}_k = \nabla u_k \qquad p_k = i\rho_0 \omega u_k$$

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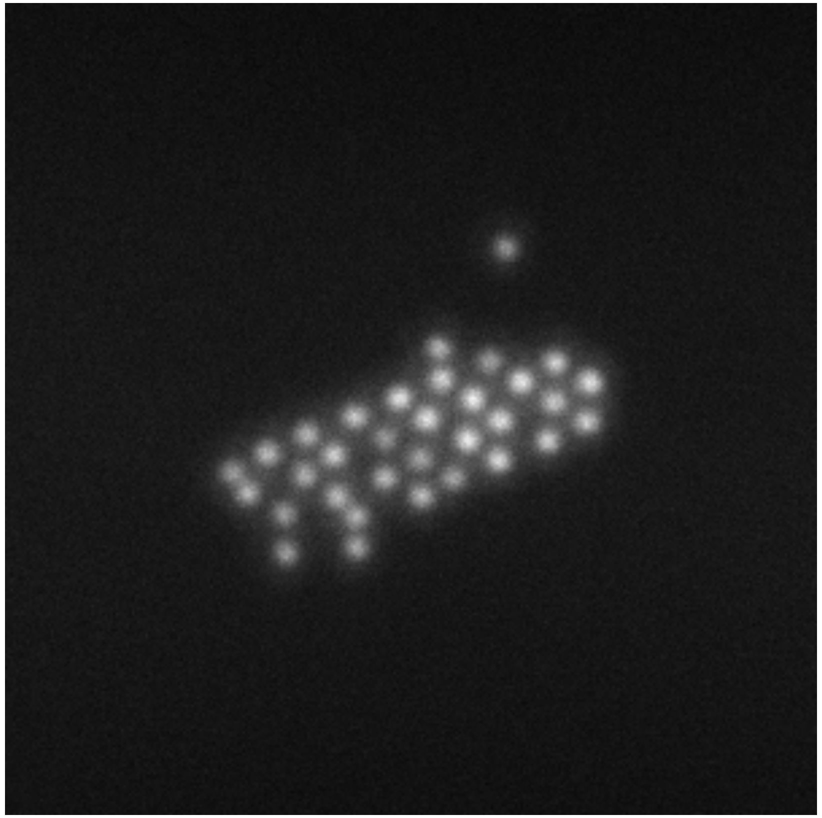


300 nm particles in an optically bound array.

MOTIVATION: SELF ORGANIZATION WITH OPTICALLY BOUND COLLOIDS

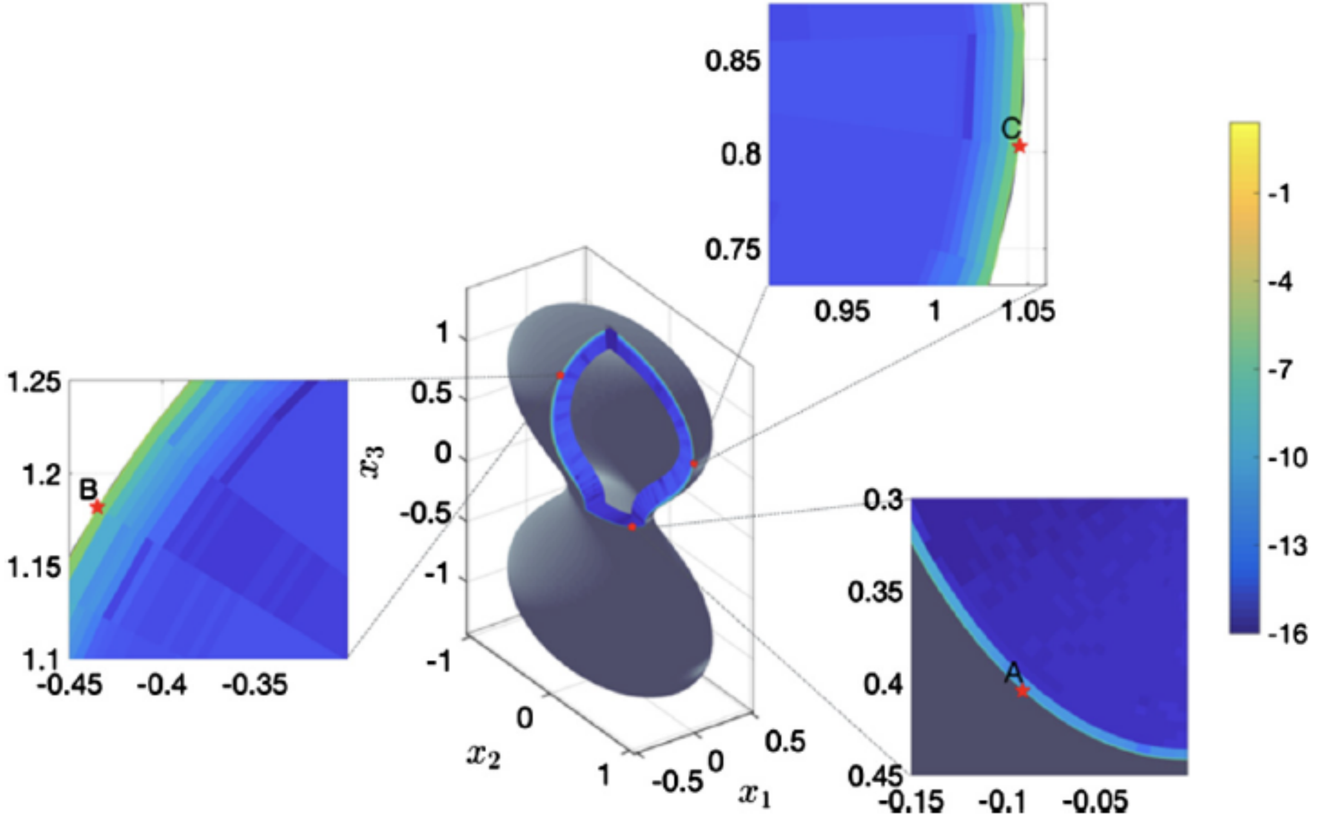
Optical Binding

- Intense laser is used to create a force between individual particles.
- Highly tunable.
- Particles arrange in interesting geometries.

Acoustic Binding

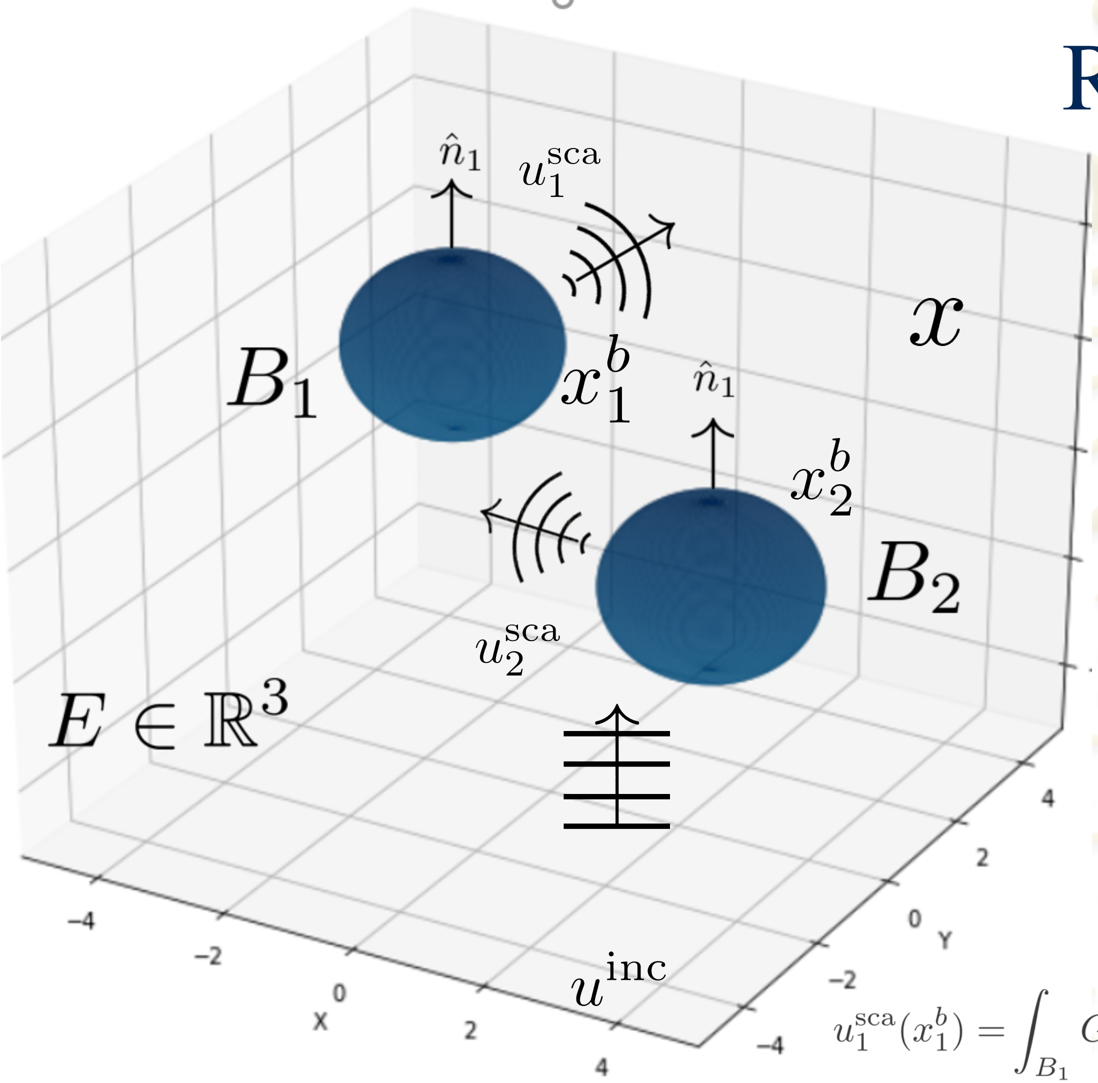
- Analogous to optical binding
- Applications in self-assembling materials.

SOLVING FOR THE SCATTERED FIELD USING A SYSTEM OF BOUNDARY INTEGRAL EQUATIONS (BIEs).



Khatri et. al. *Journal of Computational Physics 423 (2020) 109798*

Representation Formula (a brief look)



$$u^{\text{sca}} = \int_{B_1} G(x, y_1)\partial_{n_{y_1}} u^{\text{inc}}(y_1) + \partial_{n_{y_1}} G(x, y_1) u_1^{\text{sca}}(y_1)\, d\sigma_{y_1}$$

$$+ \int_{\partial B_2} G(x, y_2)\partial_{n_{y_2}} u^{\text{inc}}(y_2) + \partial_{n_{y_2}} G(x, y_2) u_2^{\text{sca}}(y_2)\, \sigma_{y_2}, \quad x \in E$$

Where, $G(x, y) = \dfrac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x, y \in \mathbb{R}^3.$

A few skipped steps (they don't fit on this slide),

and we arrive at…

$$u_1^{\text{sca}}(x_1^b) = \int_{B_1} G(x_1^b, y_1)\partial_{n_{y_1}} u^{\text{inc}}(y_1) + \partial_{n_{y_1}} G(x_1^b, y_1) u_1^{\text{sca}}(y_1)\, d\sigma_{y_1} + \frac{u_1^{\text{sca}}(x_1^b)}{2}$$

$$+ \int_{\partial B_2} G(x_1^b, y_2)\partial_{n_{y_2}} u^{\text{inc}}(y_2) + \partial_{n_{y_2}} G(x_1^b, y_2) u_1^{\text{sca}}(y_2)\, d\sigma_{y_2}, \quad x_1^b \in B_1.$$

$$u_2^{\text{sca}}(x_2^b) = \int_{B_2} G(x_2^b, y_1)\partial_{n_{y_1}} u^{\text{inc}}(y_1) + \partial_{n_{y_1}} G(x_2^b, y_1) u_1^{\text{sca}}(y_1)\, d\sigma_{y_1}$$

$$+ \int_{\partial B_2} G(x_2^b, y_2)\partial_{n_{y_2}} u_1^{\text{inc}}(y_2) + \partial_{n_{y_2}} G(x_2^b, y_2) u^{\text{sca}}(y_2)\, d\sigma_{y_2} + \frac{u_2^{\text{sca}}(x_2^b)}{2}, \quad x_2^b \in B_2.$$

LAYER POTENTIAL INTEGRAL OPERATORS (SIMPLIFYING NOTATION)

BIE System

$$u_1^{\text{sca}}(x_1^b) = \int_{B_1} G(x_1^b, y_1)\partial_{n_{y_1}} u^{\text{inc}}(y_1) + \partial_{n_{y_1}} G(x_1^b, y_1) u_1^{\text{sca}}(y_1)\, d\sigma_{y_1} + \int_{\partial B_2} G(x_1^b, y_2)\partial_{n_{y_2}} u^{\text{inc}}(y_2) + \partial_{n_{y_2}} G(x_1^b, y_2) u_1^{\text{sca}}(y_2)\, d\sigma_{y_2} + \frac{u_1^{\text{sca}}(x_1^b)}{2}, \quad x_1^b \in B_1.$$

$$u_2^{\text{sca}}(x_2^b) = \int_{B_2} G(x_2^b, y_1)\partial_{n_{y_1}} u^{\text{inc}}(y_1) + \partial_{n_{y_1}} G(x_2^b, y_1) u_1^{\text{sca}}(y_1)\, d\sigma_{y_1} + \int_{\partial B_2} G(x_2^b, y_2)\partial_{n_{y_2}} u_1^{\text{inc}}(y_2) + \partial_{n_{y_2}} G(x_2^b, y_2) u^{\text{sca}}(y_2)\, d\sigma_{y_2} + \frac{u_2^{\text{sca}}(x_2^b)}{2}, \quad x_2^b \in B_2.$$

Operator Notation

$$u_1^{\text{sca}}(x_1^b) = \mathcal{S}_{y_1}[\partial_{n_{y_1}} u^{\text{inc}}](x_1^b) + \mathcal{D}_{y_1}[u_1^{\text{sca}}](x_1^b) + \mathcal{S}_{y_2}[\partial_{n_{y_2}} u^{\text{inc}}](x_1^b) + \mathcal{D}_{y_2}[u_2^{\text{sca}}](x_1^b) + \frac{u_1^{\text{sca}}(x_1^b)}{2}$$

$$u_2^{\text{sca}}(x_2^b) = \mathcal{S}_{y_1}[\partial_{n_{y_1}} u^{\text{inc}}](x_2^b) + \mathcal{D}_{y_1}[u_1^{\text{sca}}](x_2^b) + \mathcal{S}_{y_2}[\partial_{n_{y_2}} u^{\text{inc}}](x_2^b) + \mathcal{D}_{y_2}[u_2^{\text{sca}}](x_2^b) + \frac{u_2^{\text{sca}}(x_2^b)}{2}$$

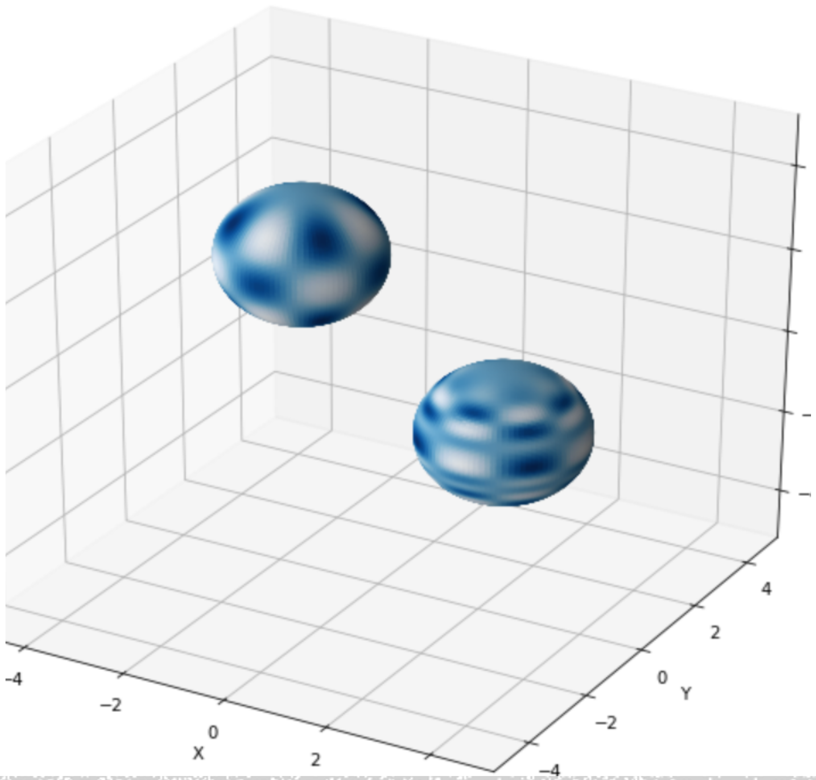
BIE System (operator notation)

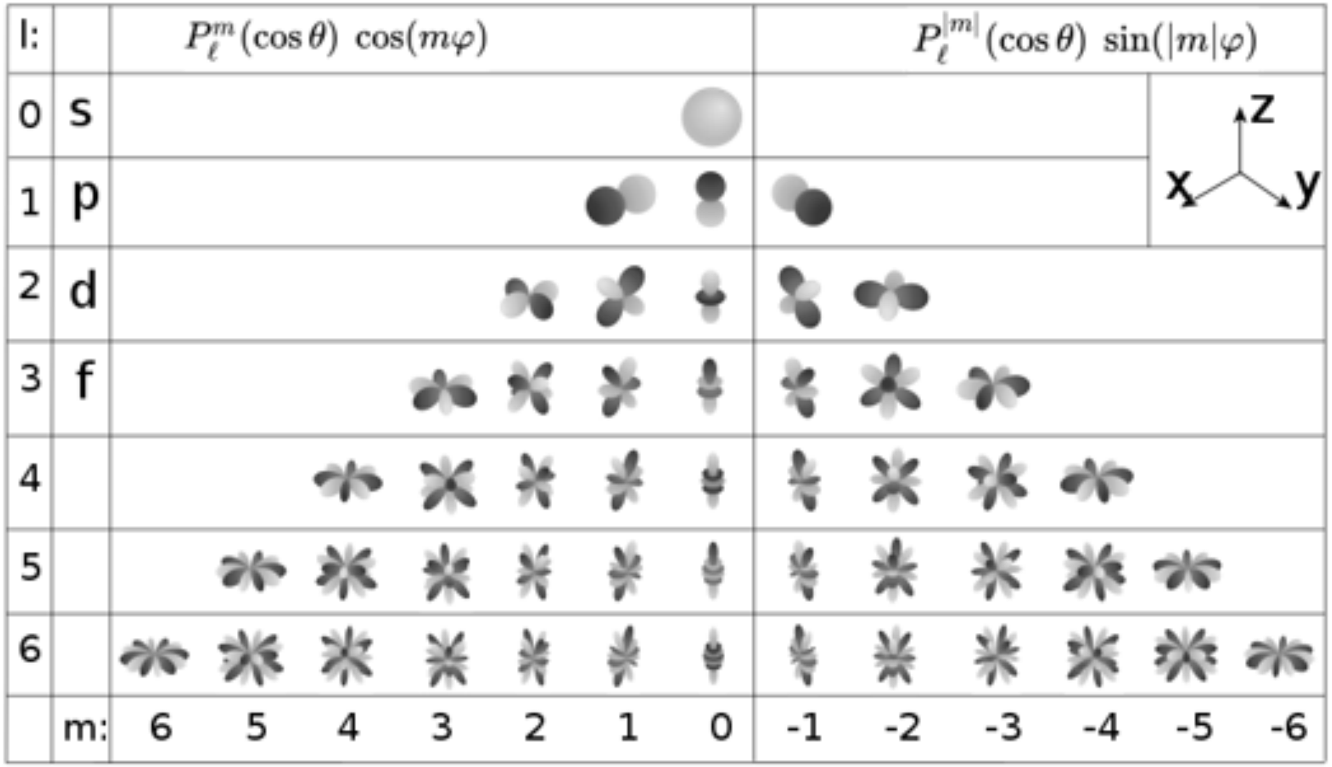
$$\frac{1}{2}u_1^{\text{sca}}(x_1^b) - \mathcal{D}_{y_1}[u_1^{\text{sca}}](x_1^b) - \mathcal{D}_{y_2}[u_2^{\text{sca}}](x_1^b) = \mathcal{S}_{y_1}[\partial_{n_{y_1}} u^{\text{inc}}](x_1^b) + \mathcal{S}_{y_2}[\partial_{n_{y_2}} u^{\text{inc}}](x_1^b)$$

$$\frac{1}{2}u_2^{\text{sca}}(x_2^b) - \mathcal{D}_{y_2}[u_2^{\text{sca}}](x_2^b) - \mathcal{D}_{y_1}[u_1^{\text{sca}}](x_2^b) = \mathcal{S}_{y_2}[\partial_{n_{y_2}} u^{\text{inc}}](x_2^b) + \mathcal{S}_{y_1}[\partial_{n_{y_1}} u^{\text{inc}}](x_2^b)$$

Diagonal Blocks

Off-Diagonal Blocks





SPHERICAL HARMONICS: EXPLOITING THE GEOMETRY OF THE PROBLEM

- Form a complete set of orthonormal functions.
- Form a basis for expansion of functions in 3-D (3-D analog of Fourier Series).
- Allow for major simplification of boundary integral equations (spoiler alert).

HARMONIC EXPANSIONS

$$G(r,\theta,\phi,r',\theta',\phi') = ik\sum_{l,m} j_l(kr)h_{l'}^{(1)}(kr')Y_l^m(\theta,\phi)\overline{Y_{m'}^{l'}(\theta',\phi')}$$

$$\frac{\partial G}{\partial n_y}(r,\theta,\phi,r',\theta',\phi') = ik\sum_{l,m} j_l(kr)[\partial_{r'}h_{l'}^{(1)}(kr')]Y_l^m(\theta,\phi)\overline{Y_{m'}^{l'}(\theta',\phi')}$$

Fundamental Solution (Green's Function)

$$u_1^{\text{sca}}(\hat{x}_1) = \sum_{l,m} C_{lm}^{(1)}h_l^{(1)}(kr_1)Y_l^m(\theta_1,\phi_1)$$

$$u_2^{\text{sca}}(\hat{x}_2) = \sum_{l,m} C_{lm}^{(2)}h_l^{(1)}(kr_2)Y_l^m(\theta_2,\phi_2)$$

Scattered Field

$$u_1^{\text{inc}}(\hat{x}_1) = \sum_{l,m} b_{lm}^{(1)}j_l(kr_1)Y_l^m(\theta_1,\phi_1)$$

$$u_2^{\text{inc}}(\hat{x}_2) = \sum_{l,m} b_{lm}^{(2)}j_l(kr_2)Y_l^m(\theta_2,\phi_2)$$

Incident Field

DIAGONAL AND OFF DIAGONAL LAYER POTENTIALS

Single-Layer

$$\int_{B_k} G(x, y_k)\partial_{n_{y_k}} u^{\text{inc}}(y_k)\, d\sigma_{y_k} = \mathcal{S}_{y_k}[\partial_{n_{y_k}} u^{\text{inc}}](x), \quad k = 1, 2$$

Double-Layer

$$\int_{B_k} \partial_{n_{y_k}} G(x, y_k) u_k^{\text{sca}}(y_k)\, d\sigma_{y_k} = \mathcal{D}_{y_k}[u_k^{\text{sca}}](x), \quad k = 1, 2$$

Layer potential operators applied to spherical harmonics

$$\mathcal{S}[Y_l^m](\theta, \phi) = ika^2 j_l(ka) h_l^{(1)}(ka) Y_l^m(\theta, \phi). \qquad \mathcal{D}[Y_l^m](\theta, \phi) = ika^2 \partial_r j_l(ka) h_l^{(1)}(ka) Y_l^m(\theta, \phi)$$

Vico et. al. [1]

Recall harmonic expansions

$$G(r, \theta, \phi, r', \theta', \phi') = ik \sum_{l,m} j_l(kr) h_{l'}^{(1)}(kr') Y_l^m(\theta, \phi) \overline{Y_{m'}^{l'}(\theta', \phi')}$$

$$\frac{\partial G}{\partial n_y}(r, \theta, \phi, r', \theta', \phi') = ik \sum_{l,m} j_l(kr) [\partial_{r'} h_{l'}^{(1)}(kr')] Y_l^m(\theta, \phi) \overline{Y_{m'}^{l'}(\theta', \phi')}$$

$$u_j^{\text{sca}}(x_j^b) = \sum_{l,m} C_{lm}^{(j)} h_l^{(1)}(kr_j) Y_l^m(\theta_j, \phi_j) j = 1, 2$$

$$u_j^{\text{inc}}(x_j^b) = \sum_{l,m} b_{lm}^{(j)} j_l(kr_j) Y_l^m(\theta_j, \phi_j)$$

DIAGONAL AND OFF DIAGONAL LAYER POTENTIALS (CONTINUED)

Diagonal Blocks

$$\mathcal{D}_{y_j}[u_j^{\text{sca}}](\hat{x}_j) = ik^2a^2 \sum_{l,m} C_{lm}^{(j)} \partial_r j_l(ka) h_l^{(1)}(ka) Y_l^m(\theta_j, \phi_j)$$

$$\mathcal{S}_{y_j}[\partial_{y_j} u_j^{\text{inc}}](\hat{x}_j) = ik^2a^2 \sum_{l,m} b_{lm}^{(j)} \partial_r j_l(ka) j_l(ka) h_l^{(1)}(ka) Y_l^m(\theta_j, \phi_j)$$

$$j = 1, 2$$

Need to find a way to evaluate a function "radiating" from sphere j onto sphere i.

Off-Diagonal Blocks

(here is where we run into trouble)

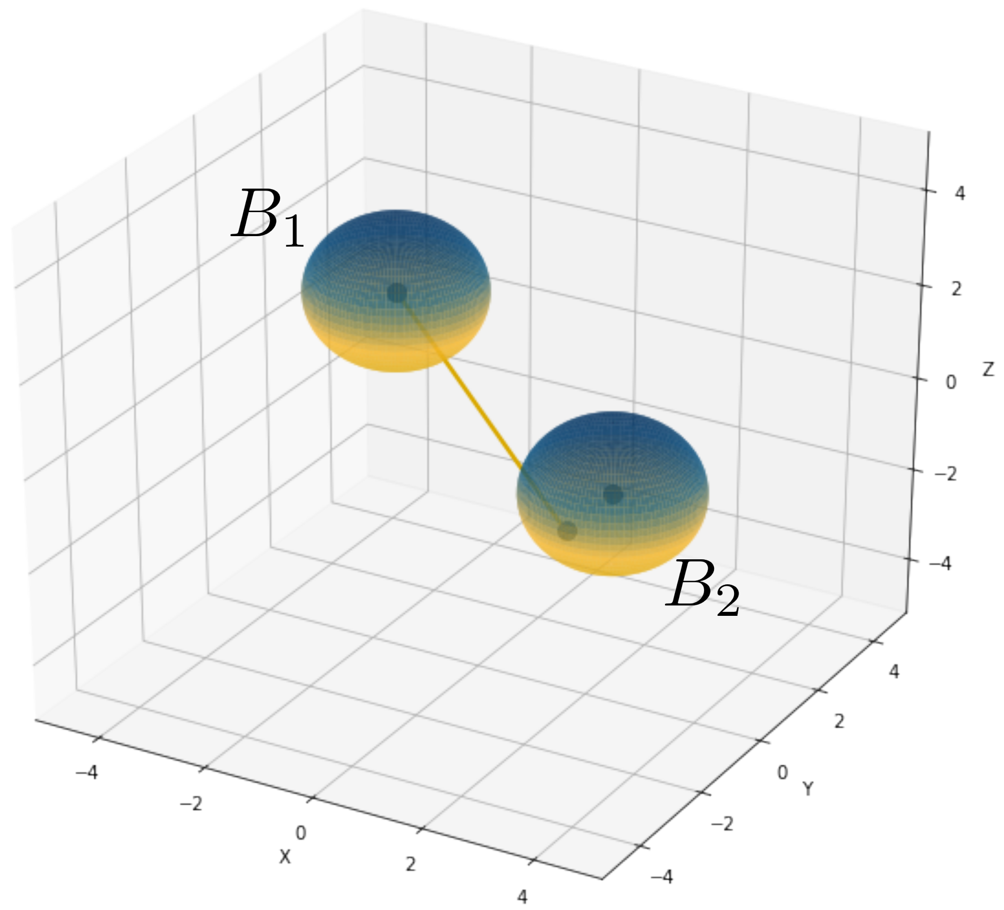
$$\mathcal{D}_{y_i}[u_i^{\text{sca}}](x_j^b) = ia^2k^2 \sum_{l,m} C_{lm}^{(i)} [\partial_r h_l^{(1)}(ka)] h_l^{(1)}(ka) j_l(kr_j(\theta_i, \phi_i)) Y_l^m(\theta_j, \phi_j)$$

$$\mathcal{S}_{y_i}[\partial_{n_{y_i}} u_i^{\text{inc}}](x_j^b) = ia^2k^2 \sum_{l,m} b_{lm}^{(i)} h_l^{(1)}(ka) [\partial_r j_l(ka)] j_l(kr_j(\theta_i, \phi_i)) Y_l^m(\theta_j, \phi_j)$$

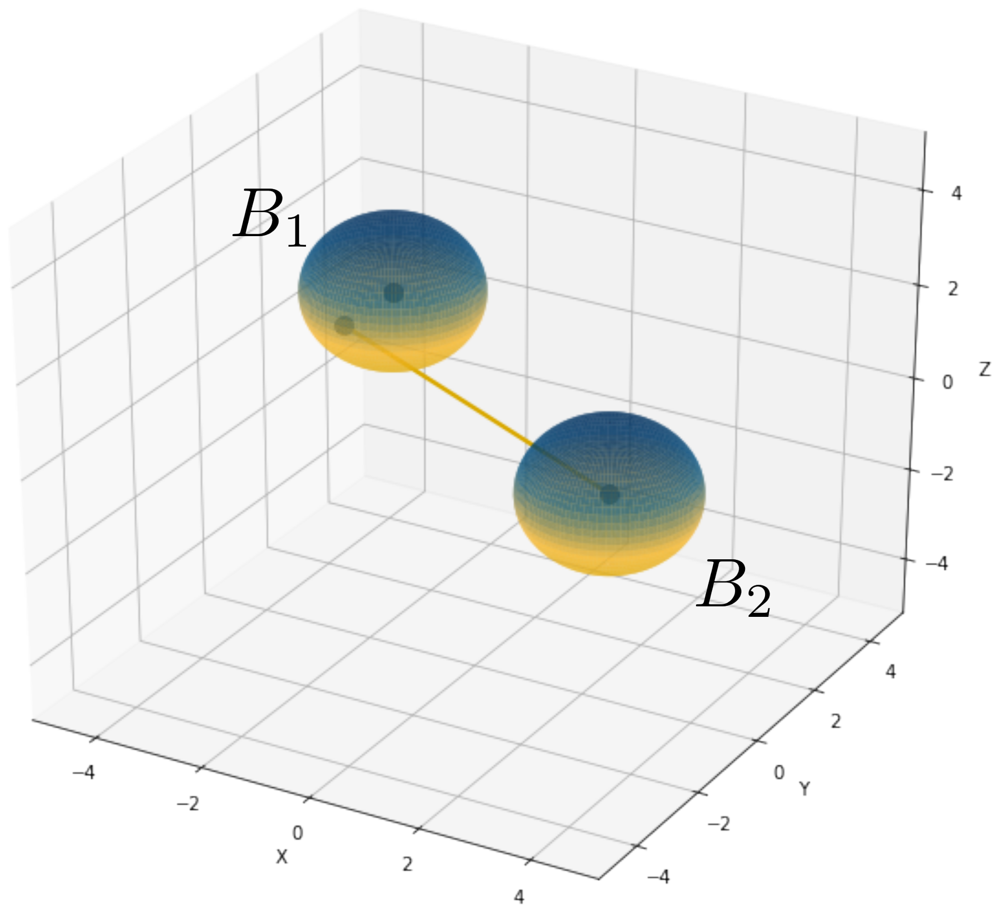
$$i, j = 1, 2, \quad i \neq j$$

VISUALIZING THE ISSUE

Mapping 1 onto 2

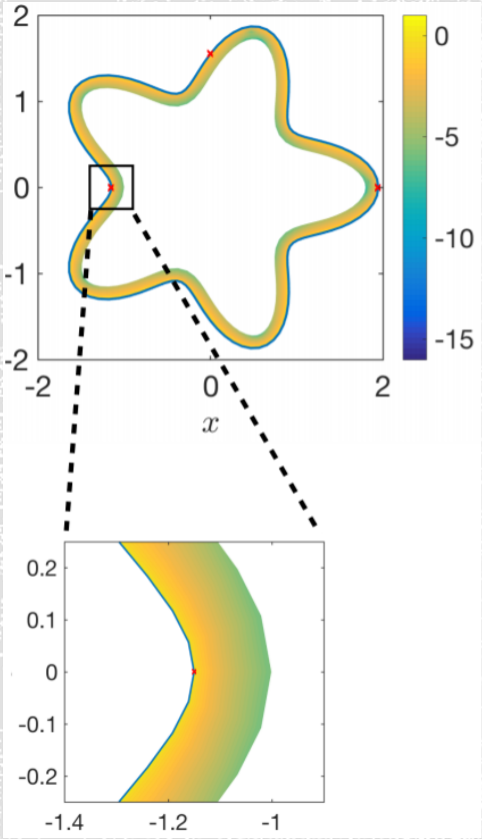


Mapping 2 onto 1



THE CLOSE EVALUATION PROBLEM

- Large error resulting from computing solution near the boundary using high order quadrature rule at fixed order.
- Non-uniform error.



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QUESTIONS